ISRAEL JOURNAL OF MATHEMATICS **169** (2009), 341–373
DOI: 10.1007/s11856-009-0014-0

ON THE DISTRIBUTION OF SATAKE PARAMETERS OF GL_2 HOLOMORPHIC CUSPIDAL REPRESENTATIONS

BY

Charles Li

Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

ABSTRACT

We prove that for a fixed non-archimedean place v of a totally real number field F, the traces of the associated Langlands classes of holomorphic cuspidal representations of $\mathrm{GL}_2(\mathbf{A})$ with trivial central character and of prime levels is equidistributed with respect to the measure

$$d\mu_v(x) = \frac{q_v + 1}{(q_v^{1/2} + q_v^{-1/2})^2 - x^2} d\mu_\infty(x),$$

where q_v is the norm of the prime ideal corresponding to v and $d\mu_\infty(x) = \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx$ is the Sato-Tate measure. This generalizes a result of Sarnak [Sa] on the distribution of Hecke eigenvalues of modular forms. The proof involves establishing a trace formula for the Hecke operators. While not explicit, this trace formula can be used as a starting point for generalizing the Eichler-Selberg trace formula to totally real number fields.

1. Introduction

Let (X, μ) be a topological measure space. Let $S_1, S_2, \ldots, S_i, \ldots$ be a sequence of finite multisets with elements in X. Let $|S_i|$ be the cardinality of S_i. We say $\{S_i\}$ is equidistributed with respect to $d\mu$ (or μ-equidistributed) if for any continuous function f on X, we have

$$\lim_{i\to\infty} \frac{\sum_{x\in S_i} f(x)}{|S_i|} = \int_X f(x)d\mu(x). \tag{1}$$

Received November 26, 2006 and in revised form June 11, 2007

If $a_1, a_2, \ldots$ is a sequence of points of X, we say that $\{a_n\}$ is μ-equidistributed if $\{S_i\}$ is μ-equidistributed where $S_i = \{a_1, \ldots, a_i\}$ (see [Se1]).

Let F be a number field. Let $\mathbf{A}$ be the adele ring of F. If π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbf{A})$, for each place v at which π is unramified, let $A_v(\pi)$ denote the associated Langlands class in $\mathrm{GL}_2(\mathbf{C})$, which is represented by a diagonal matrix $\begin{pmatrix} \alpha_{1v} & \\ & \alpha_{2v} \end{pmatrix}$. Let $\lambda_v(\pi)$ denote the trace of this class. The numbers α_{1v}, α_{2v} can also be called Satake parameters. The general Ramanujan conjecture predicts that $|\alpha_{iv}| = 1$, or equivalently, $|\lambda_v(\pi)| \leq 2$. The proof of the conjecture for GL_2 holomorphic cuspidal representations over $\mathbf{Q}$ can be found in Deligne [De] and Deligne-Serre [DS]. Brylinski and Labesse [BL, Theorem 3.4.6] showed that the conjecture is true at almost all primes for GL_2 holomorphic cuspidal representations over a totally real field F. Recently, the full conjecture (at all places, when all weights are ≥ 2) was proved by Blasius [Bl], with a parity condition on the weights. The parity requirement was removed in the thesis of his student L. Nguyen [Ng].

If the central character of π is trivial and π is a genuine cuspidal representation ([Sh, Definition 3.3]), the Sato-Tate conjecture predicts that the set $\{\lambda_v(\pi) : v \text{ with } \pi_v \text{ unramified}\}$ is equidistributed with respect to the Sato-Tate measure

$$d\mu_\infty(x) = \begin{cases} \frac{1}{\pi}\sqrt{1 - \frac{x^2}{4}}dx & \text{if } x \in [-2, 2], \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

See [Sh], [Se2, Chapter 1] and [Mu, Lecture 1]. A recent major breakthrough can be found in Taylor [Ta]. He proved the Sate-Tate conjecture for automorphic representation π of $\mathrm{GL}_2(\mathbf{A})$ for which π_v is a weight 2 discrete series representation for all the archimedean valuations v and for which the local component at some finite place is an unramified twist of the Steinberg representation.

In this paper, we do not prove these conjectures, which are still open in general. Instead, we use an idea introduced by Sarnak [Sa] for finding the distributions of Hecke eigenvalues for fixed v as levels vary. See also [Se1] and [CDF].

From now on we assume F is totally real and has degree $r \geq 2$ over $\mathbf{Q}$. Let $\mathrm{N}_{F/\mathbf{Q}}$ be the norm map. Let $\sigma_1, \ldots, \sigma_r$ be the embeddings of F into $\mathbf{R}$. Let $\infty_1, \ldots, \infty_r$ be the corresponding valuations. Let σ be the map from F to $\mathbf{R}^r$ defined by $\sigma(x) = (\sigma_1(x), \ldots, \sigma_r(x))$. Let $\mathcal{O}$ be the ring of integers. For a nonzero integral ideal $\mathfrak{a}$, let $\mathbb{N}(\mathfrak{a}) = |\mathcal{O}/\mathfrak{a}|$ denote the absolute norm. Let $\widehat{\mathcal{O}}$

denote $\prod_{v<\infty}\mathcal{O}_v$, where $v<\infty$ is the set of non-archimedean valuations. Let $\mathbf{A}_{\text{fin}}$ be the set of finite adeles. We freely identify the set with $1_\infty\times\mathbf{A}_{\text{fin}}\subset\mathbf{A}$.

Let $G=\mathrm{GL}_2$, and let Z denote its center. We freely identify $Z(\mathbf{A})$ with $\mathbf{A}^*$ throughout this paper. Let $\overline{G}$ denote $Z\backslash G$. Generally, if S is a subset of G, $\overline{S}$ denotes the image of S in $Z\backslash G$. Let $K_{\infty_i}=\left\{k_\theta=\left(\begin{smallmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{smallmatrix}\right)\right\}$ be a compact subgroup of $G(F_{\infty_i})=G(\mathbf{R})$. Let $K_\infty=\prod_{i=1}^r K_{\infty_i}$. Let $K_v=\mathrm{GL}_2(\mathcal{O}_v)$ be the standard maximal compact subgroup of $G(F_v)$. Let $K_{\text{fin}}=\mathrm{GL}_2(\widehat{\mathcal{O}})=\prod_{v<\infty}K_v$ be the standard maximal compact subgroup of $G(\mathbf{A}_{\text{fin}})$. Let $K=K_\infty K_{\text{fin}}$.

Let $L^2=L^2(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))$ be the space of square integrable functions on $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$. Let R denote the right regular representation of $G(\mathbf{A})$ on L^2. Denote by L_0^2 the subspace of cuspidal functions. The restriction of R to L_0^2 is denoted by R_0. We know that R_0 decomposes as a discrete sum of irreducible representations (see [GGPS, Chapter 3, Section 4.6]). These are the cuspidal representations. Every cuspidal representation π can be factorized as a restricted tensor product of irreducible admissible local representations $\hat{\bigotimes}\pi_v$ with π_v unramified almost everywhere (see [Fl]).

Let $\mathfrak{N}$ be an integral ideal. Let v be a non-archimedean valuation. Define groups

$$K_0(\mathfrak{N}_v)=\left\{\begin{pmatrix}a & b\\ c & d\end{pmatrix}\in K_v : c\in\mathfrak{N}_v\right\},\quad K_0(\mathfrak{N})=\prod_{v<\infty}K_0(\mathfrak{N}_v).$$

These are open compact subgroups of $G(F_v)$ and $G(\mathbf{A}_{\text{fin}})$, respectively. Let $\mathtt{k}=(\mathtt{k}_1,\ldots,\mathtt{k}_r)$ be an r-tuple of integers, each greater than 2. Let $\Pi_{\mathtt{k}}(\mathfrak{N})$ be the set of cuspidal representations π in L_0^2 for which:

1. $\pi_{\text{fin}}=\hat{\bigotimes}_{v<\infty}\pi_v$ contains a nonzero $K_0(\mathfrak{N})$-fixed vector;
2. $\pi_{\infty_i}=\pi_{\mathtt{k}_i}$ (the discrete series representation of $G(\mathbf{R})$ of weight $\mathtt{k}_i$, see [KL, Chapter 11]), for $i=1,\ldots,r$.

The set of such π is finite (see [BJ, Section 1]). Because the central character is trivial, we can assume $\mathtt{k}_1,\ldots,\mathtt{k}_r$ are even numbers.

Theorem 1.1: *Let v be a non-archimedean valuation of F and let $\mathfrak{p}$ be its corresponding prime ideal. Let $q_v=\mathbb{N}(\mathfrak{p})$. Let $\{\mathfrak{N}_i\}$ be a sequence of prime ideals different from $\mathfrak{p}$. Suppose $\mathfrak{N}_i\to\infty$ when $i\to\infty$; here $\mathfrak{N}\to\infty$ means $\mathbb{N}(\mathfrak{N})\to\infty$. Then the family of multisets $S_i=\{\lambda_v(\pi):\pi\in\Pi_{\mathtt{k}}(\mathfrak{N}_i)\}$ is* *equidistributed with respect to the measure*

$$d\mu_v(x) = \frac{q_v + 1}{(q_v^{1/2} + q_v^{-1/2})^2 - x^2} d\mu_\infty(x). \tag{3}$$

The theorem is a generalization of [Se1, Theorem 2]. The proof of this theorem is given in Section 4. As in [Se1], the proof relies on a trace formula for Hecke operators. The trace formula is interesting on its own and is given in Theorem 3.21. We closely follow [KL]. This trace formula can be a starting point for generalizing the Eichler-Selberg trace formula to Hilbert modular forms.

Here are some interesting applications of Theorem 1.1 from [Se1].

Corollary 1.2: *Let $I \subset \mathbf{R}$ be an interval. Then the proportion of $\lambda_v(\pi) \in I$ among $\{\lambda_v(\pi) : \pi \in \Pi_{\mathtt{k}}(\mathfrak{N}_i)\}$ tends to $\int_I d\mu_v(x)$ when $i \to \infty$.*

Proof. Let f be the characteristic function of I in (1). ∎

Corollary 1.3: $\{\lambda_v(\pi) : \pi \in \Pi_{\mathtt{k}}(\mathfrak{N}_i)\}$ *is dense in* $[-2, 2]$.

Proof. This follows from the fact that $\frac{d\mu_v(x)}{dx}$ is a strictly positive function on $[-2, 2]$. ∎

This shows that the bound 2 given in the Ramanujan conjecture is optimal, because there are infinitely many $\lambda_v(\pi)$ in the interval $[2 - \varepsilon, 2]$ for any $\varepsilon > 0$.

I would like to thank Andrew Knightly and Jean-Pierre Serre for their comments.

2. Test functions

Let $\pi_{\infty_i} = \pi_{\mathtt{k}_i}$ be the discrete series representation of $G(\mathbf{R})$ with weight $\mathtt{k}_i$. When π is a discrete series representation, denote by w_π a lowest weight vector of π with norm one. Define

$$A_{\mathtt{k}}(\mathfrak{N}) = \bigoplus_{\pi \in \Pi_{\mathtt{k}}(\mathfrak{N})} \mathbf{C} w_{\pi_{\infty_1}} \otimes \cdots \otimes w_{\pi_{\infty_r}} \otimes \pi_{\mathrm{fin}}^{K_0(\mathfrak{N})}, \tag{4}$$

where $\pi_{\mathrm{fin}}^{K_0(\mathfrak{N})}$ is the subspace of $K_0(\mathfrak{N})$-fixed vectors in the space of π_{fin}.

The absolute value on F_v is defined by the relation $d(ax) = |a|dx$, where dx is a Haar measure on F_v. The Haar measure on $\mathbf{A}$ is normalized such that $\mathrm{meas}(F\backslash\mathbf{A}) = 1$. For $v < \infty$, the Haar measure on F_v^* is normalized such that $\mathrm{meas}(\mathcal{O}_v^*) = 1$. The Haar measure on $G(F_v)$ is normalized such that $\mathrm{meas}(K_v) = 1$.

Let π be a representation of a group G. Let f be an element in $L^1(G)$. For w in π, define $\pi(f)w = \int_G f(g)\pi(g)wdg$. In this section we construct a function f such that the operator on L^2 defined by

$$R(f)\phi(x) = \int_{\overline{G}(\mathbf{A})} f(g)\phi(xg)dg \tag{5}$$

has finite rank ($\leq \dim A_{\mathfrak{k}}(\mathfrak{N})$), and acts like a Hecke operator on $A_{\mathfrak{k}}(\mathfrak{N})$.

Let M be the diagonal subgroup of G. Let $N = \{\left(\begin{smallmatrix} 1 & * \\ & 1 \end{smallmatrix}\right)\}$ be the unipotent subgroup. Let $B = MN$ be the set of upper triangular matrices. Define a homomorphism $H : M \to \mathbf{R}$ by $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mapsto \log \left|\frac{a}{b}\right|$. We can extend H to a function on $G(\mathbf{A})$ by $H(g) = H(m)$ if $g = mnk, m \in M(\mathbf{A}), n \in N(\mathbf{A})$ and $k \in K$. This is the height function on $G(\mathbf{A})$.

2.1. Construction of test functions. Let f be a function on $\overline{G}(\mathbf{A})$. We assume f is a product of local functions on $G(F_v)$:

$$f = f_\infty f_{\mathrm{fin}} = \prod_{i=1}^{r} f_{\infty_i} \prod_{v<\infty} f_v,$$

where f_v is the characteristic function of $\overline{K}_v$ for almost every v.

For the non-archimedean places, we take f_v to be a bi-$K_0(\mathfrak{N}_v)$-invariant function whose support is compact modulo $Z(F_v)$. We now specify the local factors of f more precisely. Let $\mathfrak{n}, \mathfrak{N}$ be two integral ideals. We further assume that $(\mathfrak{n}, \mathfrak{N}) = 1$. Let R be a ring. Denote by $M_2(R)$ the set of 2 by 2 matrices over R. For fixed v, define a set

$$M(\mathfrak{n}_v, \mathfrak{N}_v) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathcal{O}_v) : \, c \in \mathfrak{N}_v, \, (\det(g)) = \mathfrak{n}_v \right\}.$$

Let χ_v be the characteristic function of $\overline{M(\mathfrak{n}_v, \mathfrak{N}_v)}$. Define

$$f_v = \psi(\mathfrak{N}_v)\chi_v. \tag{6}$$

Here (see [KL, Lemma 13.1])

$$\begin{aligned} \psi(\mathfrak{N}_v) &= \mathrm{meas}(\overline{K_0(\mathfrak{N}_v)})^{-1} = [K_v : K_0(\mathfrak{N}_v)] \\ &= \begin{cases} q_v^{\mathrm{ord}_v(\mathfrak{N})-1}(q_v+1) & \text{if } \mathrm{ord}_v(\mathfrak{N}) > 0 \\ 1 & \text{if } \mathrm{ord}_v(\mathfrak{N}) = 0. \end{cases} \end{aligned}$$

The factor $\psi(\mathfrak{N}_v)$ in f_v is to ensure that when $v \nmid \mathfrak{n}$, $\pi_v(f_v)w = w$ if w is a $K_0(\mathfrak{N}_v)$-fixed vector. Define $\psi(\mathfrak{N}) = \prod_{v<\infty} \psi(\mathfrak{N}_v)$. Write

$$f^{\mathfrak{n}} = f^{\mathfrak{n}}_{\mathfrak{N}} = \prod_{v<\infty} f_v.$$

The notation $\mathfrak{N}$ in $f^{\mathfrak{n}}_{\mathfrak{N}}$ is usually dropped when it is clear from the context.

Let k be an integer. Let π_k be the discrete series representation of $G(\mathbf{R})$ of weight k. Denote by w_k a lowest weight vector for π_k with norm one. Let d_k be the formal degree of π_k. Let $f_k(g)$ be the normalized matrix coefficient $d_k\overline{\langle \pi_k(g)w_k, w_k\rangle}$. Explicitly, if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then ([KL, Section 14])

$$f_k(g) = \begin{cases} \frac{(k-1)}{4\pi}\frac{\det(g)^{k/2}(2i)^k}{(-b+c+(a+d)i)^k} & \text{if } \det(g) > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

This function is absolutely integrable if and only if $k > 2$. Define

$$f_{\infty_i} = f_{\mathtt{k}_i}, \quad f_{\mathtt{k}} = \prod_{i=1}^{r} f_{\mathtt{k}_i}.$$

It is easily seen that when $\det(g) > 0$,

$$|f_{\infty_i}(g) = |f_{\mathtt{k}_i}(g)| = \frac{(\mathtt{k}_i - 1)}{4\pi}\frac{2^{\mathtt{k}_i}(\det g)^{\mathtt{k}_i/2}}{(a^2+b^2+c^2+d^2+2|\det g|)^{\mathtt{k}_i/2}}. \tag{8}$$

The adjoint of $R(f)$ is $R(f)^* = R(f^*)$, where $f^*(g) = \overline{f(g^{-1})}$. A simple calculation shows that when $f = f_{\mathtt{k}}f^{\mathfrak{n}}$, $f^* = f$. Therefore, $R(f)$ is a self-adjoint operator.

Unless otherwise stated, we assume $f = f_{\mathtt{k}}f^{\mathfrak{n}}$ throughout this paper.

2.2. Image of $R(f)$.

Lemma 2.1: *Let $k > 2$ be an integer. Let $g_1, g_2 \in G(\mathbf{R})$. Then*

$$\int_{N(\mathbf{R})} f_k(g_1 n g_2)dn = 0.$$

Proof. Write $n = \left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)$,

$$f_k(g_1 n g_2) = \frac{C}{(At+B)^k},$$

where A, B and C are complex numbers depending on g_1, g_2. Because $f_k(g_1 n g_2)$ is finite as a function of $n \in N(\mathbf{R})$, $\frac{C}{(At+B)^k}$ has no poles on $\mathbf{R}$. By the residue theorem, it is easily shown that $\int_{-\infty}^{\infty} \frac{C dt}{(At+B)^k} = 0$. ∎

Proposition 2.2: *Suppose $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$. Then $R(f)$ maps*

$$A_{\mathfrak{k}}(\mathfrak{N}) \to A_{\mathfrak{k}}(\mathfrak{N})$$

and vanishes on $A_{\mathfrak{k}}(\mathfrak{N})^{\perp}$, the orthogonal complement of $A_{\mathfrak{k}}(\mathfrak{N})$ in L^2.

Proof. ([KL, p. 216–p. 219]) Since $R(f)$ is a self-adjoint operator, $R(f)$ annihilates $\mathrm{Im}(R(f))^{\perp}$: suppose $w \in \mathrm{Im}(R(f))^{\perp}$, then for any $u \in L^2$, $0 = \langle R(f)u, w\rangle = \langle u, R(f)w\rangle$. Therefore, $R(f)w$ must be 0.

To prove the proposition, it suffices to show that $\mathrm{Im}\, R(f) \subset A_{\mathfrak{k}}(\mathfrak{N})$.

First, suppose $\phi \in L^2$ is bounded, so that $|\phi(g)| \le M$ for some M. For any $g \in G(\mathbf{A})$, the constant term of $R(f)\phi$ is given by

$$\int_{N(F)\backslash N(\mathbf{A})} R(f)\phi(ng)\, dn = \int_{N(F)\backslash N(\mathbf{A})} \left[\int_{\overline{G}(\mathbf{A})} f(x)\phi(ngx)\, dx\right] dn.$$

This double integral is absolutely convergent since

$$\int_{N(F)\backslash N(\mathbf{A})} \int_{\overline{G}(\mathbf{A})} |f(x)\phi(ngx)|\, dx\, dn \le M \,\mathrm{meas}(N(F)\backslash N(\mathbf{A}))\|f\|_1 < \infty.$$

Therefore, by Fubini's theorem,

$$\begin{aligned}
&\int_{N(F)\backslash N(\mathbf{A})} R(f)\phi(ng)\, dn \\
&\qquad = \int_{N(F)\backslash N(\mathbf{A})} \left[\int_{\overline{G}(\mathbf{A})} f(g^{-1}n^{-1}x)\phi(x)\, dx\right] dn \\
&\qquad = \int_{N(F)\backslash N(\mathbf{A})} \left[\int_{N(F)\backslash \overline{G}(\mathbf{A})} \sum_{\delta \in N(F)} f(g^{-1}n^{-1}\delta x)\phi(x)\, dx\right] dn \\
&\qquad = \int_{N(F)\backslash \overline{G}(\mathbf{A})} \left[\int_{N(\mathbf{A})} f(g^{-1}nx)\, dn\right] \phi(x)\, dx = 0
\end{aligned}$$

by Lemma 2.1. Thus $R(f)\phi$ is cuspidal for bounded $\phi \in L^2$. Such functions are dense in L^2. Because $R(f)$ is a continuous operator and the cuspidal subspace is closed, $R(f)\phi$ is cuspidal for all $\phi \in L^2(\omega)$. This shows that $\mathrm{Im}\, R(f) \subset L_0^2$. Hence, $R(f)$ annihilates ${L_0^2}^{\perp}$.

It remains to show that for $w \in L_0^2$, $R(f)w \in A_{\mathfrak{k}}(\mathfrak{N})$. Because L_0^2 is the closure of the direct sum of cuspidal representations, it suffices to prove that $R(f)w \in A_{\mathfrak{k}}(\mathfrak{N})$ for any w in a cuspidal representation π. It is known that

$\pi = \pi_{\infty_1} \otimes \cdots \otimes \pi_{\infty_r} \otimes \pi_{\mathrm{fin}}$. Without loss of generality, we can take $w = w_1 \otimes \cdots \otimes w_r \otimes w_{\mathrm{fin}}$. Then

$$R(f)w = \left(\bigotimes_{i=1}^{r} \pi_{\infty_i}(f_{\infty_i}) w_i \right) \otimes \pi_{\mathrm{fin}}(f_{\mathrm{fin}}) w_{\mathrm{fin}}. \tag{9}$$

By the orthogonality properties of matrix coefficients (see [KL, Corollary 10.26] for the strong version of Schur orthogonality we use here), $\pi_{\infty_i}(f_{\infty_i})w_i = 0$ unless $\pi_{\infty_i} \cong \pi_{\mathtt{k}_i}$. Assuming $\pi_{\infty_i} \cong \pi_{\mathtt{k}_i}$, we have $\pi_{\infty_i}(f_{\infty_i})w_i \in \mathbf{C}w_{\pi_{\infty_i}}$. Because f_{fin} is left $\overline{K_0(\mathfrak{N})}$-invariant, $R(f_{\mathrm{fin}})w_{\mathrm{fin}}$ is $\overline{K_0(\mathfrak{N})}$-invariant. This completes the proof. ∎

2.3. γ's that appear in the geometric side.

Lemma 2.3: *Suppose f_v is defined as in Section 2.1. Suppose $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G(F_v)$ and that $(\det(g)) = \mathfrak{n}_v$. Then $f_v(g) \neq 0$ if and only if $g \in M_2(\mathcal{O}_v)$ and $c \in \mathfrak{N}_v$, i.e., if and only if $g \in M(\mathfrak{n}_v, \mathfrak{N}_v)$.*

Proof. $f_v(g) \neq 0$ if and only if $g = zm$, with $z \in Z(F_v)$, $m \in M(\mathfrak{n}_v, \mathfrak{N}_v)$. Taking determinants we see that z is a unit in $\mathcal{O}_v$ (identifying $Z(F_v)$ with F_v^*). Thus z can be absorbed into m, so $g \in M(\mathfrak{n}_v, \mathfrak{N}_v)$ as required. ∎

If $\mathfrak{a}$ is a fractional ideal of F, use $[\mathfrak{a}]$ to represent the corresponding ideal class in the ideal class group of F. Consider the following equation in the ideal class group of F:

$$1 = [\mathfrak{b}]^2[\mathfrak{n}]. \tag{10}$$

Suppose $[\mathfrak{b}_1], \ldots, [\mathfrak{b}_t]$ are the solutions of the equation. We can assume that $\mathfrak{b}_1, \ldots, \mathfrak{b}_t$ are integral ideals. Let $\mathtt{n}_i \in \mathcal{O}$ be a generator of $\mathfrak{b}_i^2\mathfrak{n}$, i.e.

$$\mathtt{n}_i\mathcal{O} = \mathfrak{b}_i^2\mathfrak{n}. \tag{11}$$

By Dirichlet's unit theorem, $\mathcal{O}^*$ is an abelian group of the form $\mathbf{Z}/2\mathbf{Z} \times \mathbf{Z}^{r-1}$. Obviously $\mathcal{O}^*/\mathcal{O}^{*2}$ has order 2^r. Let

$$U = \{u_1, \ldots, u_{2^r}\} \subset \mathcal{O}^* \tag{12}$$

be a fixed set of representatives for $\mathcal{O}^*/\mathcal{O}^{*2}$.

Proposition 2.4: *Suppose $\gamma \in G(F)$, $x, y \in G(\mathbf{A}_{\mathrm{fin}})$ such that $\det x^{-1}y \in \widehat{\mathcal{O}}^*$ and $f^{\mathfrak{n}}(x^{-1}\gamma y) \neq 0$. Then there exist i, j and $s \in F^*$, such that*

$$\det \gamma = s^2 \mathtt{n}_i u_j. \tag{13}$$

Here i, j are uniquely determined by γ, and s is uniquely determined up to ± 1.

Let $\tilde{\gamma} = s^{-1}\gamma \in \gamma Z(F)$. Then

$$\det \tilde{\gamma} = \mathtt{n}_i u_j. \tag{14}$$

$f^{\mathfrak{n}}(x^{-1}\gamma y) \neq 0$ if and only if we have

$$\beta^{-1}x^{-1}\tilde{\gamma}y \in \prod_{v<\infty} \left\{ g_v = \begin{pmatrix} a_v & b_v \\ c_v & d_v \end{pmatrix} \in M_2(\mathcal{O}_v) : c_v \in \mathfrak{N}_v \right\}, \tag{15}$$

where $\beta = (\beta_v)$ is in $\mathbf{A}_{\mathrm{fin}}^*$ such that β_v is a generator of $\mathfrak{b}_{iv}$.

Proof. If $f^{\mathfrak{n}}(x^{-1}\gamma y) \neq 0$, then $x_v^{-1}\gamma y_v \in Z(F_v)M(\mathfrak{n}_v, \mathfrak{N}_v)$. Taking determinants on both sides, $(\det \gamma)_v = z_v^2 \mathfrak{n}_v$ for some $z_v \in F_v^*$. Equivalently, $\mathrm{ord}_v \det \gamma \equiv \mathrm{ord}_v \mathfrak{n} \pmod 2$. Therefore, there exists an ideal $\mathfrak{b}$ such that $(\det \gamma) = \mathfrak{b}^2\mathfrak{n}$. The ideal class $[\mathfrak{b}]$ satisfies (10). By unique factorization of ideals, $\mathfrak{b}$ is uniquely determined by γ. We know that $[\mathfrak{b}] = [\mathfrak{b}_i]$ for some i. Thus there exists $h \in F^*$ such that $h\mathfrak{b} = \mathfrak{b}_i$. We have $(h^2 \det \gamma) = h^2\mathfrak{b}^2\mathfrak{n} = \mathfrak{b}_i^2\mathfrak{n} = (\mathtt{n}_i)$. Therefore, $h^2 \det \gamma = \mathtt{n}_i u$ for some $u \in \mathcal{O}^*$. Write $u = w^2 u_j$ with $w \in \mathcal{O}^*$ and $u_j \in U$ as in (12). Because h is unique up to multiplication by units, u is unique up to multiplication by squares of units. Therefore, u_j is uniquely determined by γ. Now $(hw^{-1})^2 \det \gamma = \mathtt{n}_i u_j$. Letting $s = h^{-1}w$, (13) holds. It is clear from the proof that s is unique up to ± 1.

When $\tilde{\gamma} = s^{-1}\gamma$, $\det \tilde{\gamma} = \mathtt{n}_i u_j$, thus $(\det \tilde{\gamma}) = (\mathtt{n}_i) = \mathfrak{b}_i^2\mathfrak{n}$. Localizing at v, we have $(\det \beta_v^{-1}\tilde{\gamma}) = \mathfrak{n}_v$, i.e., $(\det \beta_v^{-1}x_v^{-1}\tilde{\gamma}y_v) = \mathfrak{n}_v$. The proposition then follows easily from the previous proposition. ∎

Corollary 2.5: *Suppose $\gamma \in G(F)$, $x, y \in G(\mathbf{A}_{\mathrm{fin}})$ such that $\det x^{-1}y \in \widehat{\mathcal{O}}^*$ and $f^{\mathfrak{n}}(x^{-1}\gamma y) \neq 0$. Then there exists $\tilde{\gamma} \in G(F)$ such that $\tilde{\gamma}Z(F) = \gamma Z(F)$ and*

$$\det \tilde{\gamma} = \mathtt{n}_i u_j.$$

Here, i, j are uniquely determined by γ and $\tilde{\gamma}$ is unique up to ± 1. Also, we have

$$x^{-1}\tilde{\gamma}y \in \prod_{v<\infty} M_2(\mathcal{O}_v).$$

Proof. This follows easily from the previous proposition, (15) and the fact that $\mathfrak{b}_i$ is an integral ideal. ∎

Let $\mathtt{n} \in \mathcal{O}$. Let Q be a positive integer. Define

$$D(\mathtt{n}, Q) = \{\gamma \in Q^{-1}M_2(\mathcal{O})/\pm : \det\gamma = \mathtt{n}\}.$$

Corollary 2.6: *Let $K'_{\rm fin} = \prod_{v<\infty} K'_v$ be a compact subset of $G(\mathbf{A}_{\rm fin})$. Then there exists a positive integer Q depending only on $K'_{\rm fin}$ such that if $\gamma \in G(F)$ and $x \in K'_{\rm fin}$ satisfies $f^{\mathtt{n}}(x^{-1}\gamma x) \neq 0$, there exist unique i, j and unique $\tilde\gamma \in D(\mathtt{n}_i u_j, Q)$ with $\tilde\gamma Z(F) = \gamma Z(F)$.*

Proof. From the previous corollary, there exist unique $i, j, \tilde\gamma \in G(F)/\pm$ such that $\det\tilde\gamma = \mathtt{n}_i u_j$ and $\tilde\gamma Z(F) = \gamma Z(F)$. Because $K'_{\rm fin}$ is compact, there exists a positive integer S such that $K'_{\rm fin}$ and ${K'_{\rm fin}}^{-1} \subset S^{-1}M_2(\widehat{\mathcal{O}})$. Again from the previous corollary, $\tilde\gamma \in \prod_{v<\infty} x_v M_2(\mathcal{O}_v)x_v^{-1} \subset S^{-2}M_2(\widehat{\mathcal{O}})$. Therefore, $\tilde\gamma \in Q^{-1}M_2(\mathcal{O})$ with $Q = S^2$. ∎

3. Trace formula

In this section, we introduce Arthur's trace formula as in [Ge], [Ar1] and [Ar2]. In Arthur's work, the test function is compactly supported. However the function $f = f_{\mathtt{k}} f^{\mathtt{n}}$ we constructed is not compactly supported. So extra care should be taken about the convergence issues. The final formulas and the proofs for hyperbolic conjugacy classes and the unipotent conjugacy class are not much different from that of compactly supported test functions. We closely follow [KL, Chapters 15–22]. Here are few important properties of $f = f_{\mathtt{k}} f^{\mathtt{n}}$:

1. $f^{\mathtt{n}}$ is bounded and compactly supported modulo $Z(\mathbf{A}_{\rm fin})$. This property allows us to replace sums over $G(F)$ with discrete sums over matrices.
2. f_{∞_i} has polynomial decay (see (8)).
3. $R(f)$ is a finite rank operator.

3.1. Siegel sets. Let us recall some properties about Siegel domains ([Go, Theorem 7]). Let $\mathbf{A}^1$ be the set of norm one elements in $\mathbf{A}$. Let C_1 be a compact subset of $\mathbf{A}$. Let C_2 be a compact subset of $\mathbf{A}^1$. Let $Y > 0$ be a real number. A Siegel domain $\mathfrak{S}$ is a set of points in the form

$$\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix}\begin{pmatrix} m & \\ & 1 \end{pmatrix}k, \tag{16}$$

where $x \in C_1$, $m \in C_2$, $k \in K$ and a real number $y > Y$. The number y can be identified with an idele whose infinite components are y and whose finite components are 1.

When $\mathfrak{S}$ is sufficiently large, i.e., C_1 and C_2 are sufficiently large and Y is sufficiently small, we have $\overline{G}(\mathbf{A}) = \overline{G}(F)\mathfrak{S}$.

Let Y' be a positive number and C' be a compact subset of $\mathbf{R}$. Let $\mathcal{D}$ be the set:

$$\left\{\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{1/2} & \\ & y^{-1/2} \end{pmatrix} : x \in C', y > Y' \right\}.$$

By absorbing the infinite component of m into the y-part and the finite component of m into k in (16), we obtain a variant of the Siegel domain:

$$\mathfrak{S}' = \prod_{i=1}^{r} \mathcal{D} K_{\infty_i} \times \prod_{v<\infty} K'_v, \tag{17}$$

where K'_v is an open compact set and is equal to K_v for a.e. v. Define $K'_{\infty_i} = K_{\infty_i}$. Let $K' = \prod_v K'_v$. When C' and K' are sufficiently large and Y' is sufficiently small, we have $\mathfrak{S} \subset \mathfrak{S}'$. Let $\mathfrak{S}'_\infty = \prod_{i=1}^r \mathcal{D} K_{\infty_i}$.

Let C_{01} be a compact subset of $\mathbf{A}$. Let C_{02} be a compact subset of $\mathbf{A}^1$. Let $\mathfrak{S}_0$ be a set of points in the form of (16) with $x \in C_{01}$, $m \in C_{02}$, $k \in K$ and real number $y > 0$. When C_{01}, C_{02} are sufficiently large, we have $\overline{G}(\mathbf{A}) = \overline{B}(F)\mathfrak{S}_0$ and $\mathfrak{S} \subset \mathfrak{S}_0$.

From now on, we assume $\mathfrak{S}$, $\mathfrak{S}_0$, $\mathfrak{S}'$ and $K' = \prod K'_v$ are fixed as above.

3.2. Kernel functions. Let f be a continuous function of $G(\mathbf{A})$. The kernel of $R(f)$ is formally defined as

$$K(g_1, g_2) = \sum_{\gamma \in \overline{G}(F)} f(g_1^{-1}\gamma g_2). \tag{18}$$

When f is a compactly supported function, the kernel function is a finite sum when g_1, g_2 belong to compact sets. We are going to prove the absolute convergence and continuity of the kernel function for $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$.

Let $\alpha, \beta \in L^1(G)$. The convolution of α and β is given by

$$(\alpha * \beta)(x) = \int_G \alpha(g)\beta(g^{-1}x)dg = \int_G \alpha(xg)\beta(g^{-1})dg.$$

This product makes $L^1(G)$ into an associative algebra.

For an integrable function $f \in L^1(\overline{G}(\mathbf{A}))$, we give a condition under which the kernel $K(g_1, g_2) = \sum_\gamma f(g_1^{-1}\gamma g_2)$ is absolutely convergent and continuous in both variables. The idea behind the proposition is from [Os, p. 86].

PROPOSITION 3.1: *Let $C_1, C_2 \subset G(\mathbf{A})$ be subsets with compact image in $\overline{G}(\mathbf{A})$. Suppose ψ_1, ψ_2 are bounded measurable compactly supported functions on $\overline{G}(\mathbf{A})$. Suppose $f \in L^1(\overline{G}(\mathbf{A}))$ satisfies*

$$f = \psi_1 * f = f * \psi_2. \tag{19}$$

Then for every $\gamma \in \overline{G}(F)$, there exists a real number α_γ, independent of $g_1 \in C_1$ and $g_2 \in C_2$, such that

$$|f(g_1^{-1}\gamma g_2)| \leq \alpha_\gamma$$

and

$$\sum_{\gamma \in \overline{G}(F)} \alpha_\gamma < \infty.$$

Proof. Let M be an upper bound for $|\psi_1|$ and $|\psi_2|$. Then for any $(g_1, g_2) \in C_1 \times C_2$,

$$\begin{aligned} |f(g_1^{-1}\gamma g_2)| = |(\psi_1 * f)(g_1^{-1}\gamma g_2)| &= \left| \int_{\overline{G}(\mathbf{A})} \psi_1(h_1) f(h_1^{-1} g_1^{-1}\gamma g_2) dh_1 \right| \\ &= \left| \int_{\overline{G}(\mathbf{A})} \psi_1(g_1^{-1}h_1) f(h_1^{-1}\gamma g_2) dh_1 \right| \\ &\leq M \int_{\overline{g_1 \operatorname{Supp} \psi_1}} |f(h_1^{-1}\gamma g_2)| dh_1. \end{aligned}$$

Similarly,

$$|f(h_1^{-1}\gamma g_2)| \leq M \int_{\overline{g_2 (\operatorname{Supp} \psi_2)^{-1}}} |f(h_1^{-1}\gamma h_2)| dh_2.$$

Thus we can choose

$$\alpha_\gamma = M^2 \int_{\overline{C_1 \operatorname{Supp} \psi_1}} \int_{\overline{C_2 (\operatorname{Supp} \psi_2)^{-1}}} |f(h_1^{-1}\gamma h_2)| dh_2 dh_1.$$

Let $\mathcal{B} \subset \overline{G}(\mathbf{A})$ be a fundamental domain for $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$. Let $\mathcal{B}'$ be an open set, slightly bigger than $\mathcal{B}$. We assume that $\mathcal{B}'$ can be covered by finitely many translations of $\mathcal{B}$. Because $\overline{C_2 (\operatorname{Supp} \psi_2)^{-1}}$ is compact, it can be covered by finitely many open sets of the form $\gamma\mathcal{B}'$ for $\gamma \in \overline{G}(F)$. Hence there exist

$\gamma_1, \gamma_2, \ldots, \gamma_m \subset \overline{G}(F)$ such that $\gamma_1\mathcal{B} \cup \cdots \cup \gamma_m\mathcal{B}$ covers $\overline{C_2(\operatorname{Supp}\psi_2)^{-1}}$. We have now

$$\begin{aligned}\sum_{\gamma\in\overline{G}(F)} \alpha_\gamma &\leq \sum_{\gamma\in\overline{G}(F)} M^2 \int_{\overline{C_1 \operatorname{Supp}\psi_1}} \sum_{i=1}^{m} \int_{\gamma_i\mathcal{B}} |f(h_1^{-1}\gamma h_2)| dh_2 dh_1 \\ &\leq mM^2 \int_{\overline{C_1 \operatorname{Supp}\psi_1}} \int_{\mathcal{B}} \sum_{\gamma\in\overline{G}(F)} |f(h_1^{-1}\gamma h_2)| dh_2 dh_1 \\ &= mM^2 \int_{\overline{C_1 \operatorname{Supp}\psi_1}} \int_{\overline{G}(\mathbf{A})} |f(h_1^{-1} h_2)| dh_2 dh_1 \\ &= mM^2 \operatorname{meas}(\overline{C_1 \operatorname{Supp}\psi_1}) \|f\|_1 < \infty. \qquad \blacksquare\end{aligned}$$

Proposition 3.2: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$. Suppose Γ is a subset of $\overline{G}(F)$, then*

$$\sum_{\gamma\in\Gamma} f(g_1^{-1}\gamma g_2)$$

is absolutely convergent and defines a continuous function in both variables.

Proof. Because $f = f_{\mathtt{k}} f^{\mathfrak{n}}$ is a continuous function in $L^1(\overline{G}(\mathbf{A}))$ and $G(\mathbf{A})$ is locally compact, it suffices to find appropriate ψ_1 and ψ_2 as above. We construct $\psi = \psi_1$ first.

Define $\psi_{\text{fin}} = \frac{1}{\operatorname{meas}(\overline{K_0(\mathfrak{N})})} \chi_{\overline{K_0(\mathfrak{N})}}$. Because $f^{\mathfrak{n}}$ is left-$K_0(\mathfrak{N})$-invariant, we have $\psi_{\text{fin}} * f^{\mathfrak{n}} = f^{\mathfrak{n}}$.

Let ϕ_i be a continuous, compactly supported function on $B(\mathbf{R})$ such that $\phi_i(-b) = (-1)^{\mathtt{k}_i}\phi_i(b)$. Then ϕ_i can be extended to $G(\mathbf{R})$ by

$$\tilde{\phi}_i(k_\theta b) = e^{-i\mathtt{k}_i\theta}\phi_i(b).$$

Suppose $\pi = \pi_{\mathtt{k}_i}$. Let $w_{\mathtt{k}_i} = w_\pi$ be a lowest weight vector of π with norm one. Then it is easy to show that

$$\pi(k_\theta)\pi(\tilde{\phi}_i)w_{\mathtt{k}_i} = e^{i\mathtt{k}_i\theta}\pi(\tilde{\phi}_i)w_{\mathtt{k}_i}.$$

Therefore, $\pi(\tilde{\phi}_i)w_{\mathtt{k}_i} = c_i w_{\mathtt{k}_i}$ for some $c_i \in \mathbf{C}$.

The adjoint of $\pi(\tilde{\phi}_i)$ is $\pi(\tilde{\phi}_i^*)$. Using the Iwasawa decomposition $G = BK$, we see that

$$\pi(\tilde{\phi}_i^*)w_{\mathtt{k}_i} = \int_{\overline{B}(\mathbf{R})} \overline{\phi_i(b^{-1})}\pi(b)w_{\mathtt{k}_i} db.$$

By the standard usage of Dirac sequence (see, for example, [La, Section 1.1]), we can show that the above integral can be arbitrarily close to $w_{\mathtt{k}_i}$ by suitable choices of ϕ_i. Therefore,

$$0 \neq \left\langle w_{\mathrm{k}_i}, \pi(\tilde{\phi}_i^*) w_{\mathrm{k}_i} \right\rangle = \left\langle \pi(\tilde{\phi}_i) w_{\mathrm{k}_i}, w_{\mathrm{k}_i} \right\rangle = c_i \| w_{\mathrm{k}_i} \|^2,$$

i.e., $c_i \neq 0$. Take $\psi_{\infty_i} = \tilde{\phi}_i / c_i$. Then $\pi(\psi_{\infty_i}) w_{\mathrm{k}_i} = w_{\mathrm{k}_i}$. Therefore,

$$\begin{aligned}(\psi_{\infty_i} * f_{\mathrm{k}_i})(g) &= d_{\mathrm{k}_i} \int_{\overline{G}(\mathbf{R})} \psi_{\infty_i}(h) \overline{\langle \pi(g) w_{\mathrm{k}_i}, \pi(h) w_{\mathrm{k}_i} \rangle} dh \\ &= d_{\mathrm{k}_i} \overline{\langle \pi(g) w_{\mathrm{k}_i}, \pi(\psi_{\infty_i}) w_{\mathrm{k}_i} \rangle} \\ &= d_{\mathrm{k}_i} \overline{\langle \pi(g) w_{\mathrm{k}_i}, w_{\mathrm{k}_i} \rangle} \\ &= f_{\mathrm{k}_i}(g).\end{aligned}$$

We can take $\psi_1 = \psi = (\prod_{i=1}^r \psi_{\infty_i}) \times \psi_{\text{fin}}$.

The function ψ_2 can be constructed similarly. In fact, by the self-adjointness of f, we can take $\psi_2 = \psi_1^*$. ∎

Lemma 3.3: *If $\phi \in A_{\mathrm{k}}(\mathfrak{N})$, then ϕ is a continuous function.*

Proof. From the construction of $\psi = \psi_1$ in the previous proposition, $R(\psi)\phi = \phi$. Since ϕ is cuspidal, it is well-known that $|R(\psi)\phi|$ is bounded [Bu, Proposition 3.2.3]. Let M be an upper bound.

Let V be a fixed open neighborhood of the identity in $\overline{G}(\mathbf{A})$ with compact closure. Since ψ is compactly supported and continuous, it is uniformly continuous [KL, Proposition 10.11], i.e., for any $\varepsilon > 0$, there exists an open neighborhood U of the identity such that $|\psi(y_1) - \psi(y_2)| < \varepsilon$ whenever $y_1 y_2^{-1} \in U$. Without loss of generality, we can assume $U \subset V$. For $x_1^{-1} x_2 \in U$,

$$\begin{aligned}|R(\psi)\phi(x_1) - R(\psi)\phi(x_2)| &= \left| \int_{\overline{G}(\mathbf{A})} (\psi(x_1^{-1} g) - \psi(x_2^{-1} g)) \phi(g) dg \right| \\ &\leq \int_{x_1 U \operatorname{Supp} \psi} |\psi(x_1^{-1} g) - \psi(x_2^{-1} g)| |\phi(g)| dg \\ &\leq \varepsilon M \operatorname{meas}(V \operatorname{Supp} \psi).\end{aligned}$$

Therefore, $\phi = R(\psi)\phi$ is continuous. ∎

When $f = f_{\mathrm{k}} f^{\mathfrak{n}}$, $R(f)$ annihilates $A_{\mathrm{k}}(\mathfrak{N})^{\perp}$ by Proposition 2.2. Because $A_{\mathrm{k}}(\mathfrak{N})$ is a finite dimensional space, $R(f)$ is a finite rank operator and hence is of trace class. Another way to express the kernel is

$$\Phi(g_1, g_2) = \sum_{\phi} R(f)\phi(g_1) \overline{\phi(g_2)}, \tag{20}$$

where ϕ runs through an orthonormal basis of $A_{\mathrm{k}}(\mathfrak{N})$. It is known that $\Phi(g_1, g_2) = K(g_1, g_2)$ almost everywhere (see [KL, Chapter 15]). Since any function in $A_{\mathrm{k}}(\mathfrak{N})$ is continuous, $\Phi(g_1, g_2)$ is a continuous function. By Proposition 3.2, $K(g_1, g_2)$ is also a continuous function. Therefore, $\Phi(g_1, g_2) = K(g_1, g_2)$ everywhere. One can then easily show that

$$\operatorname{tr} R(f) = \sum_{\phi} \langle R(f)\phi, \phi\rangle = \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} K(g, g)dg. \tag{21}$$

The Hecke operators $R(f_{\mathrm{k}} f^{\mathfrak{n}})$ on $A_{\mathrm{k}}(\mathfrak{N})$ for $(\mathfrak{n}, \mathfrak{N}) = 1$ are commutative and self-adjoint and thus simultaneously diagonalizable. For $\pi \in \Pi_{\mathrm{k}}(\mathfrak{N})$, let $E_{\mathrm{k}}(\pi, \mathfrak{N})$ be a basis for $\pi \cap A_{\mathrm{k}}(\mathfrak{N})$ consisting of simultaneous eigenvectors of the Hecke operators. Let $E_{\mathrm{k}}(\mathfrak{N})$ be a basis consisting of simultaneous eigenvectors of the Hecke operators. We can take $E_{\mathrm{k}}(\mathfrak{N}) = \bigcup E_{\mathrm{k}}(\pi, \mathfrak{N})$. We have

$$K(g_1, g_2) = \sum_{\pi\in\Pi_{\mathrm{k}}(\mathfrak{N})} \sum_{\varphi\in E_{\mathrm{k}}(\pi,\mathfrak{N})} \frac{R(f)\varphi(g_1)}{\|\varphi\|} \frac{\overline{\varphi(g_2)}}{\|\varphi\|}. \tag{22}$$

Define a polynomial of degree n by

$$X_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

The main properties of X_n can be found in [Se1].

Proposition 3.4: *Suppose* $\mathfrak{n} = \mathfrak{p}^n$. *Then*

$$K(g_1, g_2) = q_v^{n/2} \sum_{\pi\in\Pi_{\mathrm{k}}(\mathfrak{N})} \sum_{\varphi\in E_{\mathrm{k}}(\pi,\mathfrak{N})} X_n(\lambda_v(\pi)) \frac{\varphi(g_1)}{\|\varphi\|} \frac{\overline{\varphi(g_2)}}{\|\varphi\|}. \tag{23}$$

Proof. Let φ be a nonzero element in $\pi \cap A_{\mathrm{k}}(\mathfrak{N})$. Let ϖ_v be a uniformizer of $\mathcal{O}_v$. Let χ be the character on $B(F_v)$ defined by $\chi(\begin{smallmatrix} a & b \\ & d\end{smallmatrix}) = \chi_1(a)\chi_2(d)$, where χ_i is the unramified character given by $\chi_i(\varpi_v) = \alpha_{iv}$. Because $v \nmid \mathfrak{N}$, $\pi_v^{K_v}$ is nontrivial. It is known that π_v is isomorphic to $\operatorname{Ind}_{B(F_v)}^{G(F_v)} \chi$. Using the well-known left coset decomposition of $M(\mathfrak{n}, \mathfrak{N})_v$(see [KL, Lemma 13.4]), we can show that $R(f)\varphi = q_v^{n/2} \sum_{i=0}^n \alpha_{1v}^i \alpha_{2v}^{n-i}\varphi$. The reader is referred to [Ro, Lemma 2.8] for the case $n = 1$. Therefore, $R(f)\varphi = q_v^{n/2} X_n(\lambda_v(\pi))\varphi$. The proposition follows easily from (22). ∎

Let π_v be an irreducible admissible infinite-dimensional representation of $G(F_v)$. Let $\mathfrak{c}(\pi_v)$ be the conductor of π_v, i.e., the largest integral ideal $\mathfrak{c}_v$ of $\mathcal{O}_v$ such that $\pi_v^{K_0(\mathfrak{c}_v)}$ is nonempty. For more details we refer the reader to [Ca, Theorem 1]. When π is a cuspidal representation, $\pi = \hat{\bigotimes}\pi_v$. Define $\mathfrak{c}(\pi)$ to be largest integral ideal $\mathfrak{c}$ of $\mathcal{O}$ such that $\pi^{K_0(\mathfrak{c})}$ is nonzero. Obviously, $\mathfrak{c}(\pi) = \prod_{v<\infty} \mathfrak{p}_v^{\mathrm{ord}_v\, \mathfrak{c}(\pi_v)}$.

Let $\mathfrak{a}$ be an integral ideal. Define $d(\mathfrak{a})$ to be the number of distinct integral ideal factors of $\mathfrak{a}$. It can be calculated by

$$d(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} (\mathrm{ord}_{\mathfrak{p}}(\mathfrak{a}) + 1).$$

Proposition 3.5: *Suppose* $\mathfrak{n} = \mathfrak{p}^n$. *Then*

$$\mathrm{tr}\, R(f) = q_v^{n/2} \sum_{\pi\in\Pi_{\mathfrak{k}}(\mathfrak{N})} d(\mathfrak{N}/\mathfrak{c}(\pi)) X_n(\lambda_v(\pi)).$$

When $\pi \in \Pi_{\mathfrak{k}}(\mathfrak{N})$, $\mathfrak{c}(\pi)|\mathfrak{N}$, *therefore,* $\mathfrak{N}/\mathfrak{c}(\pi)$ *is an integral ideal and thus* $d(\mathfrak{N}/\mathfrak{c}(\pi))$ *is meaningful.*

Proof. By (21) and (23),

$$\mathrm{tr}\, R(f) = q_v^{n/2} \sum_{\pi\in\Pi_{\mathfrak{k}}(\mathfrak{N})} \sum_{\varphi\in E_{\mathfrak{k}}(\pi,\mathfrak{N})} X_n(\lambda_v(\pi)).$$

It suffices to show that $|E_{\mathfrak{k}}(\pi,\mathfrak{N})| = \dim \pi_{\mathrm{fin}}^{K_0(\mathfrak{N})} = d(\mathfrak{N}/\mathfrak{c}(\pi))$. By [Fl, Theorem 4], the space of K-finite vectors of π is in a natural way an admissible irreducible $G(\mathbf{A}_{\mathrm{fin}})$-module $\pi^{K-\mathrm{fin}}$ and $\pi^{K-\mathrm{fin}}$ can be factorized as $\bigotimes_{v<\infty} \pi_v^{K_v-\mathrm{fin}}$ with $\pi_v^{K_v-\mathrm{fin}}$ an admissible irreducible $G(F_v)$-module. By [Ca, Corollary on p. 306], if $\mathfrak{c}(\pi)|\mathfrak{N}$, $\dim \pi_{\mathrm{fin}}^{K_0(\mathfrak{N})} = \prod_{v<\infty} \dim \pi_v^{K_0(\mathfrak{N}_v)} = \prod_{v<\infty} (\mathrm{ord}_v(\mathfrak{N}_v/\mathfrak{c}(\pi_v)) + 1) = d(\mathfrak{N}/\mathfrak{c}(\pi))$. ∎

To get rid of the multiplicity $d(\mathfrak{N}/\mathfrak{c}(\pi))$, we can use the technique in [Se1, Section 5.1]. Here we state the case which is of interest to us.

Proposition 3.6: *Let* $\mathfrak{n} = \mathfrak{p}^n$. *Let* $\mathfrak{N} = \mathfrak{P}^s$, *where* $\mathfrak{P}$ *is a prime ideal and* s *is a positive integer. Let* $f' = f_{\mathfrak{k}} f_{\mathfrak{N}'}^{\mathfrak{n}}$ *be the test function corresponding to* $\mathfrak{N}' = \mathfrak{P}^{s-1}$, *then*

$$\mathrm{tr}\, R(f) - \mathrm{tr}\, R(f') = q_v^{n/2} \sum_{\pi\in\Pi_{\mathfrak{k}}(\mathfrak{N})} X_n(\lambda_v(\pi)).$$

Proof. Obviously $\Pi_{\mathrm{k}}(\mathfrak{N}') \subset \Pi_{\mathrm{k}}(\mathfrak{N})$. By the previous proposition

$$\begin{aligned}\operatorname{tr} R(f) - \operatorname{tr} R(f') =& q_v^{n/2} \sum_{\pi\in\Pi_{\mathrm{k}}(\mathfrak{N})-\Pi_{\mathrm{k}}(\mathfrak{N}')} d(\mathfrak{N}/\mathfrak{c}(\pi))X_n(\lambda_v(\pi))\\ &+ q_v^{n/2} \sum_{\pi\in\Pi_{\mathrm{k}}(\mathfrak{N}')} (d(\mathfrak{N}/\mathfrak{c}(\pi)) - d(\mathfrak{N}'/\mathfrak{c}(\pi)))X_n(\lambda_v(\pi)).\end{aligned}$$

It remains to show that all the coefficients of $X_n(\lambda_v(\pi))$ are 1. If $\pi \in \Pi_{\mathrm{k}}(\mathfrak{N}')$, $\mathfrak{c}(\pi)|\mathfrak{N}'$. We have

$$d(\mathfrak{P}^s/\mathfrak{c}(\pi)) - d(\mathfrak{P}^{s-1}/\mathfrak{c}(\pi)) = (\operatorname{ord}_{\mathfrak{P}}(\mathfrak{P}^s/\mathfrak{c}(\pi))+1) - (\operatorname{ord}_{\mathfrak{P}}(\mathfrak{P}^{s-1}/\mathfrak{c}(\pi))+1) = 1.$$

If $\pi \in \Pi_{\mathrm{k}}(\mathfrak{N}) - \Pi_{\mathrm{k}}(\mathfrak{N}')$, then $\mathfrak{c}(\pi)|\mathfrak{N}$ but $\nmid \mathfrak{N}'$. This implies $\mathfrak{c}(\pi) = \mathfrak{N}$. Therefore, $d(\mathfrak{N}/\mathfrak{c}(\pi)) = 1$, and the proposition follows. ∎

3.3. The Truncated Kernels. Let τ be the characteristic function of $[0,\infty)$. Let $T > 0$ be a sufficiently large real number. We now define Arthur's truncated kernel as found in [Ge, Lecture II Section 3]:

$$k^T(g,f) = K(g,g) - \sum_{\delta\in B(F)\backslash G(F)} \int_{N(\mathbf{A})} \sum_{\mu\in\overline{M}(F)} f(g^{-1}\delta^{-1}\mu n\delta g)dn\,\tau(H(\delta g)-T).$$

By [GJ, Lemma 5.6], $\sum_\delta$ is a finite sum for fixed g.

For $f = f_{\mathrm{k}} f^{\mathfrak{n}}$, by Proposition 3.2 and the fact that $N(F)\backslash N(\mathbf{A})$ is compact, we easily see that the integrals

$$\begin{aligned}&\int_{N(F)\backslash N(\mathbf{A})} \sum_{\eta\in N(F)} \sum_{\mu\in\overline{M}(F)} f(g^{-1}\delta^{-1}\mu\eta n\delta g)dn\\ &\qquad= \int_{N(\mathbf{A})} \sum_{\mu\in\overline{M}(F)} f(g^{-1}\delta^{-1}\mu n\delta g)dn\end{aligned}$$

are absolutely convergent. Thus $k^T(g,f)$ is well-defined.

By Lemma 2.1, when $f = f_{\mathrm{k}} f^{\mathfrak{n}}$,

$$\int_{N(\mathbf{A})} \sum_{\mu\in\overline{M}(F)} f(g^{-1}\delta^{-1}\mu n\delta g)dn = \sum_{\mu\in\overline{M}(F)} \int_{N(\mathbf{A})} f(g^{-1}\delta^{-1}\mu n\delta g)dn = 0.$$

Therefore,

$$K(g,g) = k^T(g,f)$$

and hence by (21)

$$\operatorname{tr} R(f) = \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} k^T(g,f)dx. \tag{24}$$

We partition $\overline{G}(F)$ into an equivalence relation weaker than ordinary conjugacy classes. Two elements are equivalent if their semi-simple components are conjugate in the usual sense (see [Ge, Lecture II] or [Ar1, p. 920]). We still call an equivalence class under this relation a conjugacy class.

Let $\mathfrak{o}$ be a conjugacy class in $\overline{G}(F)$. Define

$$k_{\mathfrak{o}}^{T}(g,f)$$
$$=\sum_{\gamma\in\mathfrak{o}} f(g^{-1}\gamma g)-\sum_{\delta\in B(F)\backslash G(F)}\int_{N(\mathbf{A})}\sum_{\mu\in\overline{M}(F)\cap\mathfrak{o}} f(g^{-1}\delta^{-1}\mu n\delta g)dn\,\tau(H(\delta g)-T).$$

We have

$$K(g,g)=k^{T}(g,f)=\sum_{\mathfrak{o}}k_{\mathfrak{o}}^{T}(g,f) \tag{25}$$

and we are going to show that each of these terms is absolutely integrable over $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$. Denote

$$J_{\mathfrak{o}}^{T}(f)=\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})}k_{\mathfrak{o}}^{T}(g,f)dg.$$

We say that $\gamma\in G(F)$ is elliptic when it is not conjugate to an upper triangular matrix (over F), or equivalently, when the eigenvalues of γ lie outside F. An element of $G(F)$ is hyperbolic if it has distinct F-rational eigenvalues. Such a matrix is conjugate to a diagonal matrix in $G(F)$. We say that γ is unipotent if it has a single eigenvalue, occurring as a double root of its characteristic polynomial.

The characterization of γ as elliptic, hyperbolic, or unipotent clearly depends only on the conjugacy class to which γ belongs, and is well-defined in $\overline{G}(F)$.

We say $\mathfrak{o}$ is elliptic (hyperbolic or unipotent) if $\gamma\in\mathfrak{o}$ is elliptic (hyperbolic or unipotent). There is only one unipotent conjugacy class.

Proposition 3.7: *Let $\mathfrak{n}\in\mathcal{O}-\{0\}$. Let Q be a positive integer. Then*

$$\int_{\mathfrak{S}'_{\infty}}\sum_{\substack{\gamma\notin B(F),\\ \gamma\in D(\mathfrak{n},Q)}}|f_{\mathfrak{k}}(g^{-1}\gamma g)|dg$$

is finite.

Proof. Denote by $\mathcal{F}$ a fundamental domain of $\mathbf{R}^{r}/\sigma(\mathcal{O})$. We can assume that its closure is a parallelepiped with the origin as one of the vertices. We can further assume $\mathcal{F} \subset [-M, M]^r$ for some positive number M. In what follows the constants in $\ll$ depend only on the field F, $\mathfrak{S}'_\infty$, $\mathtt{n}$, $\mathtt{k}$, Q and M.

If $\sigma_i(\mathtt{n}) < 0$ for some i, $f_{\mathtt{k}_i}(g_{\infty_i}^{-1}\sigma_i(\gamma)g_{\infty_i}) = 0$ for $\gamma \in D(\mathtt{n}, Q)$. The proposition is trivial. Therefore we assume $\sigma_i(\mathtt{n}) > 0$ for all i.

We follow the proof of [KL, Lemma 18.3]. Let $\gamma = \left(\begin{smallmatrix} a/Q & b/Q \\ c/Q & d/Q \end{smallmatrix}\right)$ with $a, b, c, d \in \mathcal{O}$. Let $\mathtt{n}' = \mathtt{n}Q^2$. We have $ad - bc = \mathtt{n}'$. Because γ is elliptic, $c \neq 0$. Obviously b is uniquely determined by a, c and d.

Below we use α^{σ_i} to represent $\sigma_i(\alpha)$ when $\alpha \in F$ or $G(F)$. By (17), $g_{\infty_i} = \left(\begin{smallmatrix} 1 & x_i \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} y_i^{1/2} & \\ & y_i^{-1/2} \end{smallmatrix}\right)k_{\theta_i} \in \mathcal{D} \times K_{\infty_i}$. It is easy to show that $|f_k(k_{\theta_1}xk_{\theta_2})| = |f_\infty(x)|$. By (8),

$$f_{\mathtt{k}_i}(g_{\infty_i}^{-1}\gamma^{\sigma_i}g_{\infty_i}) \ll \frac{1}{((a^{\sigma_i} - c^{\sigma_i}x_i)^2 + (d^{\sigma_i} + c^{\sigma_i}x_i)^2 + (Y'c^{\sigma_i})^2 + 2\mathtt{n}'^{\sigma_i})^{\mathtt{k}_i/2}}.$$

Let $\boldsymbol{\varepsilon}_\ell = (\varepsilon_{\ell 1}, \ldots, \varepsilon_{\ell r}) \in \mathcal{F}$ for $\ell = 1, 2, 3$. By the elementary inequality $\frac{1}{x^2+\Delta^2} \leq \left(1 + \left|\frac{\varepsilon}{\Delta}\right|\right)^2 \frac{1}{(x+\varepsilon)^2+\Delta^2}$ [KL, Lemma 18.1],

$$|f_{\mathtt{k}_i}(g_{\infty_i}^{-1}\gamma^{\sigma_i}g_{\infty_i})| \leq C_i E_i,$$

where

$$E_i = \frac{1}{((a^{\sigma_i} - c^{\sigma_i}x_i + \varepsilon_{1i})^2 + (d^{\sigma_i} + c^{\sigma_i}x_i + \varepsilon_{2i})^2 + (Y'(c^{\sigma_i} + \varepsilon_{3i}))^2 + 2\mathtt{n}'^{\sigma_i})^{\mathtt{k}_i/2}}.$$

and

$$C_i = \left(1 + \frac{M}{\sqrt{2\mathtt{n}'^{\sigma_i}}}\right)^{\mathtt{k}_i}\left(1 + \frac{M}{\sqrt{2\mathtt{n}'^{\sigma_i}}}\right)^{\mathtt{k}_i}\left(1 + \frac{Y'M}{\sqrt{2\mathtt{n}'^{\sigma_i}}}\right)^{\mathtt{k}_i} \ll 1.$$

Let $m_{\mathcal{F}} = \mathrm{meas}(\mathcal{F})$ and $d\boldsymbol{\varepsilon}_\ell = d\varepsilon_{\ell 1} \cdots d\varepsilon_{\ell r}$. Then

$$\begin{aligned}
&\sum_\gamma f(g^{-1}\gamma g) \\
&\quad \ll \sum_{c\in\mathcal{O}}\sum_{d\in\mathcal{O}}\sum_{a\in\mathcal{O}} \frac{1}{m_{\mathcal{F}}^3} \iiint_{\mathcal{F}^3} \left(\prod_{i=1}^r E_i\right) d\boldsymbol{\varepsilon}_1 d\boldsymbol{\varepsilon}_2 d\boldsymbol{\varepsilon}_3 \\
&\quad = \frac{1}{m_{\mathcal{F}}^3} \sum_{c\in\mathcal{O}} \int_{\boldsymbol{\varepsilon}_3\in\mathcal{F}} \int_{\mathbf{R}^r}\int_{\mathbf{R}^r} \frac{du_1\cdots du_r\, dv_1\cdots dv_r\, d\varepsilon_{31}\cdots d\varepsilon_{3r}}{\prod_{i=1}^r (u_i^2 + v_i^2 + (Y'(c^{\sigma_i} + \varepsilon_{3i}))^2 + 2\mathtt{n}'^{\sigma_i})^{\mathtt{k}_i/2}} \\
&\quad = \frac{1}{m_{\mathcal{F}}^3} \prod_{i=1}^r \int_{\mathbf{R}}\int_{\mathbf{R}}\int_{\mathbf{R}} \frac{du_i\, dv_i\, dw_i}{(u_i^2 + v_i^2 + (Y'w_i)^2 + 2\mathtt{n}'^{\sigma_i})^{\mathtt{k}_i/2}}
\end{aligned}$$

The last line follows by using spherical coordinates and the fact that $\mathtt{k}_i \geq 4$. Therefore,

$$\int_{\mathfrak{S}'_\infty} \sum_{\substack{\gamma \notin B(F), \\ \gamma \in D(\mathtt{n}, Q)}} f_{\mathtt{k}}(g^{-1}\gamma g) dg \ll \prod_{i=1}^{r} \int_{Y'}^{\infty} \int_{C'} \frac{dx_i dy_i}{y_i^2} < \infty. \qquad \blacksquare$$

Remark: Since the central character is trivial, $\mathtt{k}_i$ is an even number strictly greater than 2, i.e., $\mathtt{k}_i \geq 4$. The above proof is only valid for this case. For the case of non-trivial central characters, we have to prove the proposition for $\mathtt{k}_i \geq 3$. The technique used in the proof of [KL, Lemma 18.3] still works but a complete proof is more complicated.

Corollary 3.8: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$. Then*

$$\int_{\mathfrak{S}'} \sum_{\gamma \notin \overline{B}(F)} \psi(\mathfrak{N})^{-1} |f|(g^{-1}\gamma g) dg$$

is bounded by a constant independent of $\mathfrak{N}$.

Proof. When $g \in \mathfrak{S}'$, $g_{\text{fin}} \in K'_{\text{fin}}$. Let Q be the integer defined in Corollary 2.6. By Corollary 2.6, we can replace $\sum_{\gamma \notin \overline{B}(F)}$ by $\sum_{i,j} \sum_{\gamma \notin B(F), \gamma \in D(\mathtt{n}_i u_j, Q)}$. Let χ be the characteristic function of $\overline{M(\mathfrak{n}, \mathfrak{N})}$.

$$\begin{aligned} &\int_{\mathfrak{S}'} \sum_{\gamma \notin \overline{B}(F)} \psi(\mathfrak{N})^{-1} |f|(g^{-1}\gamma g) dg \\ &\qquad = \sum_{i,j} \int_{\mathfrak{S}'} \sum_{\substack{\gamma \notin B(F), \\ \gamma \in D(\mathtt{n}_i u_j, Q)}} |(f_{\mathtt{k}} \times \chi)(g^{-1}\gamma g)| dg \\ &\qquad \leq \operatorname{meas}(K'_{\text{fin}}) \sum_{i,j} \int_{\mathfrak{S}'_\infty} \sum_{\substack{\gamma \notin B(F), \\ \gamma \in D(\mathtt{n}_i u_j, Q)}} |f_{\mathtt{k}}|(g^{-1}\gamma g) dg. \end{aligned}$$

Then the result follows from the previous proposition. $\blacksquare$

Corollary 3.9: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$. Then*

$$\int_{\overline{G}(F) \backslash \overline{G}(\mathbf{A})} \sum_{\gamma \text{ elliptic}} |f|(g^{-1}\gamma g) dg < \infty.$$

Proof. When γ is elliptic, $\gamma \notin \overline{B}(F)$. Because $\overline{G}(\mathbf{A}) = \overline{G}(F)\mathfrak{S}'$, the corollary follows easily by the previous proposition. $\blacksquare$

Proposition 3.10: *Let* $f = f_{\mathtt{k}} f^{\mathtt{n}}$. *Let* $\mathfrak{o}$ *be hyperbolic or unipotent. If* $J_{\mathfrak{o}}^T(f) \neq 0$, *then there exists a diagonal matrix* $\mu = \begin{pmatrix} a & \\ & b \end{pmatrix} \in M_2(\mathcal{O})$ *such that* $\overline{\mu} \in \mathfrak{o}$, $\det \mu = \mathtt{n}_i u_j$ *for some* i, j *and* $\sigma_\ell(\det \mu) = \sigma_\ell(\mathtt{n}_i u_j) > 0$ *for* $\ell = 1, \ldots, r$.

Proof. If $J_{\mathfrak{o}}^T(f) \neq 0$, then either there exist $\mu \in M(F)$, $\gamma \in \overline{G}(F)$, $\eta \in N(F)$ and $g \in \overline{G}(\mathbf{A})$ such that $\overline{\mu} \in \mathfrak{o}$ and

$$f(g^{-1}\gamma^{-1}\mu\eta\gamma g) \neq 0,$$

or there exist μ as above, $\delta \in B(F)\backslash G(F)$, $n \in N(\mathbf{A})$ and $g \in \overline{G}(\mathbf{A})$ such that

$$f(g^{-1}\delta^{-1}\mu n \delta g) \neq 0.$$

By the definition of $f_{\mathtt{k}}$, either $\det(g_{\infty_\ell}^{-1}\sigma_\ell(\gamma)^{-1}\sigma_\ell(\mu\eta)\sigma_\ell(\gamma)g_{\infty_\ell}) = \sigma_\ell(\det \mu) > 0$ for all ℓ or $\det(g_{\infty_\ell}^{-1}\sigma_\ell(\delta)^{-1}\sigma_\ell(\mu)n_{\infty_\ell}\sigma_\ell(\delta)g_{\infty_\ell}) = \sigma_\ell(\det \mu) > 0$ for all ℓ. By Corollary 2.5, we can assume $\mu = \begin{pmatrix} a & \\ & b \end{pmatrix}$ satisfying $ab = \mathtt{n}_i u_j$ for some i, j and either

$$g_{\mathrm{fin}}^{-1}\gamma^{-1}\mu\eta\gamma g_{\mathrm{fin}} \in \prod_{v<\infty} M_2(\mathcal{O}_v) \text{ or } g_{\mathrm{fin}}^{-1}\delta^{-1}\mu n_{\mathrm{fin}}\delta g_{\mathrm{fin}} \in \prod_{v<\infty} M_2(\mathcal{O}_v).$$

Therefore, $\mu\eta$ (or μn_{fin}) is conjugate to an element in $\prod_{v<\infty} M_2(\mathcal{O}_v)$. As a result, the characteristic polynomial $(x-a)(x-b)$ is a monic polynomial over $\bigcap_{v<\infty} \mathcal{O}_v = \mathcal{O}$. This implies that a, b are in $\mathcal{O}$. ∎

Unlike the case $F = \mathbf{Q}$ (cf. [KL, Section 19]), the number of factorizations of $\mathtt{n}_i u_j$ into ab $(a, b \in \mathcal{O})$ is not finite. In fact, $ab = (au)(bu^{-1})$ for any $u \in \mathcal{O}^*$. In order to prove the absolute convergence of the hyperbolic terms, we need the following proposition.

Proposition 3.11: *Let* $\mu = \begin{pmatrix} a & \\ & b \end{pmatrix} \in G(F)$ *such that* $\sigma_i(\det \mu) = \sigma_i(ab) > 0$ *for* $i = 1, \ldots, r$. *Let* $u \in \mathcal{O}^*$ *and* $\mu_u = \begin{pmatrix} au & \\ & bu^{-1} \end{pmatrix}$. *Let* $\mathfrak{o}_u$ *be the conjugacy class containing* μ_u. *Let* $f = f_{\mathtt{k}} f^{\mathtt{n}}$. *Then*

$$\sum_{u \in \mathcal{O}^*/\pm 1} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} |k_{\mathfrak{o}_u}^T(g, f)| dg < \infty.$$

Remark: The conjugacy classes $\mathfrak{o}_u$ with $u \in \mathcal{O}^*/\pm 1$ may not be distinct. In fact, if $u = b/a$ is a unit, then $\mathfrak{o}_{u_1} = \mathfrak{o}_{u_2}$ if and only if $u_1 = \pm u_2$ or $u_1 u_2 = \pm b/a$.

Proof. We follow the arguments given in [KL, Theorem 19.5]. We break $\mathfrak{o}_u$ into two parts:

$$\mathfrak{o}_u' = \mathfrak{o}_u \cap \overline{B}(F) \quad \text{and} \quad \mathfrak{o}_u'' = \mathfrak{o}_u - \mathfrak{o}_u'.$$

We break $k^T_{\mathfrak{o}_u}(g,f)$ into two parts corresponding to this partition of $\mathfrak{o}_u$:

$$k^T_{\mathfrak{o}_u}(g,f) = k^T_{\mathfrak{o}'_u}(g,f) + K_{\mathfrak{o}''_u}(g,g),$$

where

$$\begin{aligned} k^T_{\mathfrak{o}'_u}(g,f) = &\sum_{\gamma\in\mathfrak{o}'_u} f(g^{-1}\gamma g) \\ &- \sum_{\delta\in B(F)\backslash G(F)}\ \sum_{\nu\in\overline{M}(F)\cap\mathfrak{o}_u} \int_{N(\mathbf{A})} f(g^{-1}\delta^{-1}\nu n\delta g)dn\,\tau(H(\delta g)-T) \end{aligned}$$

and

$$K_{\mathfrak{o}''_u}(g,g) = \sum_{\gamma\in\mathfrak{o}''_u} f(g^{-1}\gamma g).$$

Let $\mathcal{B}\subset\mathfrak{S}$ be a fundamental domain of $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$. By Corollary 3.8,

$$\sum_{u\in\mathcal{O}^*/\pm 1}\int_{\mathcal{B}} |K_{\mathfrak{o}''_u}(g,g)|dg \le 2\int_{\mathfrak{S}'}\sum_{\gamma\notin\overline{B}(F)} |f(g^{-1}\gamma g)|dg < \infty.$$

It remains to show that $\sum_{u\in\mathcal{O}^*/\pm1}\int_{\mathcal{B}}|k^T_{\mathfrak{o}'_u}(g,f)|dg<\infty$. For $\xi\in\mathbf{A}$, $\nu\in M(F)$, define

$$\Phi_{g,\nu}(\xi) = f\left(g^{-1}\nu\begin{pmatrix}1 & \xi\\ & 1\end{pmatrix}g\right).$$

For $t\in\mathbf{A}$, the Fourier transform of $\Phi_{g,\nu}$ is

$$\widehat{\Phi}_{g,\nu}(t) = \int_{\mathbf{A}} f\left(g^{-1}\nu\begin{pmatrix}1 & \xi\\ & 1\end{pmatrix}g\right)\theta(\xi t)d\xi,$$

where θ is a fixed non-trivial character on $F\backslash\mathbf{A}$. We follow the arguments given in [Ge, Lecture II.4] and [KL, Theorem 19.5]. Let $g\in\mathfrak{S}_0$ with $H(g)\ge T$. Use the notations in (16),

$$g = \begin{pmatrix}1 & x\\ & 1\end{pmatrix}\begin{pmatrix}y^{1/2} & \\ & y^{-1/2}\end{pmatrix}\begin{pmatrix}m & \\ & 1\end{pmatrix}k, \quad H(g)=\log|y|\ge T.$$

Let ℓ be an integer ≥ 2. The proposition follows if we can show that

$$\sum_{u\in\mathcal{O}^*/\pm1}\sum_{\beta\in F^*}\widehat{\Phi}_{g,\,\mu_u}(\beta)\le Ay^{-\ell},$$

where A is a constant that does not depend on g (see [KL, (19.4)]).

Let $\ell_1, \dots, \ell_r$ be integers ≥ 2. Let $t \in \mathbf{A}_\infty^*$. Below all the constants in $\ll$ do not depend on g, t or u. By [KL, Lemma 19.11],

$$\widehat{\Phi}_{g,\mu_u,\infty_i}(t_{\infty_i}) \ll \frac{1}{|t_{\infty_i}|^{\ell_i}} \int_{\mathbf{R}} |\Phi^{(\ell_i)}_{g,\mu_u,\infty_i}(s)| ds.$$

Simple calculation shows that (see [KL] proof of Proposition 19.12)

$$\begin{aligned}
&\widehat{\Phi}_{g,\mu_u,\infty_i}(t_{\infty_i}) \\
&\qquad \ll \frac{1}{|t_{\infty_i}|^{\ell_i}} \int_{\mathbf{R}} \frac{|y^{-1} m_{\infty_i}^{-1} \sigma_i(au)|^{\ell_i} ds}{((|\sigma_i(au)| + |\sigma_i(bu^{-1})|)^2 + |y^{-1} m_{\infty_i}^{-1} \sigma_i(au)|^2 s^2)^{(\mathrm{k}_i+\ell_i)/2}} \\
&\qquad \ll \frac{|y^{-1} m_{\infty_i}^{-1} \sigma_i(au)|^{\ell_i - 1}}{(|\sigma_i(au)| + |\sigma_i(bu^{-1})|)^{\mathrm{k}_i+\ell_i-1}} \frac{1}{|t_{\infty_i}|^{\ell_i}} \int_{\mathbf{R}} \frac{ds}{(1+s^2)^{(\mathrm{k}_i+\ell_i)/2}} \\
&\qquad \ll \frac{1}{(|\sigma_i(au)| + |\sigma_i(bu^{-1})|)^{\mathrm{k}_i}} \frac{1}{|t_{\infty_i}|^{\ell_i}} \frac{1}{y^{\ell_i-1}} \int_{\mathbf{R}} \frac{ds}{1+s^2} \\
&\qquad \ll \frac{1}{(|\sigma_i(u)| + |\sigma_i(u)^{-1}|)^2} \frac{1}{|t_{\infty_i}|^{\ell_i}} \frac{1}{y^{\ell_i-1}}.
\end{aligned}$$

Since $\mathfrak{S}_{0,\mathrm{fin}}$ is a compact set, it can be covered by finite cosets in the form of $xK_0(\mathfrak{N})$, $x \in G(\mathbf{A})_{\mathrm{fin}}$. We can show that $\mathrm{Supp}\, \widehat{\Phi}_{g,\mu_u,\mathrm{fin}}$ is a compact set independent of g and u by following the argument of the proof of Proposition 19.12 on p. 258 in [KL]. Suppose the support $\subset \frac{1}{M}\widehat{\mathcal{O}}$ for some positive integer M. For every $\alpha \in \mathcal{O}$, we construct an r-tuple of positive integers $(\ell_{\alpha 1}, \dots, \ell_{\alpha r})$ as follows

$$\ell_{\alpha i} = \begin{cases} \ell & \text{if } \sigma_i(\alpha) \leq 1 \\ \ell + 1 & \text{otherwise.} \end{cases}$$

In other words, $|\sigma_i(\alpha)|^{\ell_{\alpha i}} = |\sigma_i(\alpha)|^\ell \max(1, |\sigma_i(\alpha)|)$. For $\beta = \frac{\alpha}{M} \in \frac{1}{M}\mathcal{O} - \{0\}$,

$$\begin{aligned}
&|\widehat{\Phi}_{g,\mu_u}(\beta)| \\
&\ll \frac{1}{\prod_{i=1}^r (|\sigma_i(u)| + |\sigma_i(u)^{-1}|)^2} \prod_{i=1}^r \frac{M^{\ell_{\alpha i}}}{y^{\ell_{\alpha i}-1} |\sigma_i(\alpha)|^{\ell_{\alpha i}}} \\
&\leq \frac{1}{\prod_{i=1}^r (|\sigma_i(u)| + |\sigma_i(u)^{-1}|)^2} \left(\frac{1}{|\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^\ell} \frac{1}{\prod_{i=1}^r \max(1, |\sigma_i(\alpha)|)} \right) \frac{M^{r(\ell+1)}}{y^{(\ell-1)r}} \\
&\leq \frac{1}{\prod_{i=1}^r (|\sigma_i(u)| + |\sigma_i(u)^{-1}|)^2} \left(\frac{1}{|\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^\ell} \frac{1}{\prod_{i=1}^r (1 + |\sigma_i(\alpha)|)/2} \right) \frac{M^{r(\ell+1)}}{y^{(\ell-1)r}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\sum_{u\in\mathcal{O}^*/\pm 1}\sum_{\beta\in F^*}&|\widehat{\Phi}_{g\,,\mu_u}(\beta)|\\ &\ll\Bigg(\sum_{u\in\mathcal{O}^*/\pm 1}\frac{1}{\prod_{i=1}^r(|\sigma_i(u)|+|\sigma_i(u)^{-1}|)^2}\Bigg)\\ &\quad\times\Bigg(\sum_{\alpha\in\mathcal{O}-\{0\}}\frac{1}{|\,\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^{\ell}}\frac{1}{\prod_{i=1}^r(1+|\sigma_i(\alpha)|)}\Bigg)\frac{1}{y^{(\ell-1)r}}.\end{aligned}$$

The convergences of the summations over u and α are proved in the next two lemmas. Therefore

$$\sum_{u\in\mathcal{O}^*/\pm 1}\sum_{\beta\in F^*}|\widehat{\Phi}_{g\,,\mu_u}(\beta)|\ll\frac{1}{y^{(\ell-1)r}}\le\frac{1}{y^{\ell}}.$$

This completes the proof. ∎

Let $\Lambda=\{(\log|\sigma_1(u)|,\ldots,\log|\sigma_r(u)|):u\in\mathcal{O}^*\}$ be a sublattice of $L_0=\{(x_1,\ldots,x_r)\in\mathbf{R}^r:x_1+\cdots+x_r=0\}$.

Lemma 3.12:

$$\sum_{u\in\mathcal{O}^*/\pm 1}\frac{1}{\prod_{i=1}^r(|\sigma_i(u)|+|\sigma_i(u)^{-1}|)^2}<\infty.$$

Proof.

$$\begin{aligned}\sum_{u\in\mathcal{O}^*/\pm 1}\frac{1}{\prod_{i=1}^r(|\sigma_i(u)|+|\sigma_i(u)^{-1}|)^2}&=\sum_{\lambda\in\Lambda}\frac{1}{\prod_{i=1}^r(e^{\lambda_i}+e^{-\lambda_i})^2}\\ &\ll\int_{x_1+\cdots+x_r=0}\frac{dx_1\cdots dx_{r-1}}{\prod_{i=1}^r(e^{x_i}+e^{-x_i})^2}\\ &\le\int_{\mathbf{R}^{r-1}}\frac{dx_1\cdots dx_{r-1}}{\prod_{i=1}^{r-1}(e^{x_i}+e^{-x_i})^2}<\infty.\end{aligned}$$ ∎

Lemma 3.13: *Let $\ell\ge 2$ be an integer. Then*

$$\sum_{\alpha\in\mathcal{O}-\{0\}}\frac{1}{|\,\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^{\ell}}\frac{1}{\prod_{i=1}^r(1+|\sigma_i(\alpha)|)}<\infty$$

Proof. The summation is equal to

$$\begin{aligned}
&\sum_{\alpha\in\mathcal{O}-\{0\}} \frac{1}{|\,\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^{\ell}} \frac{1}{\prod_{i=1}^{r}|\sigma_i(\alpha)|^{1/2}\prod_{i=1}^{r}(|\sigma_i(\alpha)|^{1/2}+|\sigma_i(\alpha)|^{-1/2})}\\
=&2\sum_{\alpha\in\mathcal{O}^*\backslash\mathcal{O}-\{0\}} \frac{1}{|\,\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^{\ell+1/2}} \sum_{\lambda\in\Lambda}\frac{1}{\prod_{i=1}^{r}(e^{(\lambda_i+\log|\sigma_i(\alpha)|)/2}+e^{-(\lambda_i+\log|\sigma_i(\alpha)|)/2})}\\
\ll&\sum_{\alpha\in\mathcal{O}^*\backslash\mathcal{O}-\{0\}} \frac{1}{|\,\mathrm{N}_{F/\mathbf{Q}}(\alpha)|^{\ell+1/2}} \int_{L_0}\frac{dx_1\cdots dx_{r-1}}{\prod_{i=1}^{r}(e^{(\log|\sigma_i(\alpha)|+x_i)/2}+e^{-(\log|\sigma_i(\alpha)|+x_i)/2})}\\
\ll&\sum_{\text{principal ideals }(\alpha)} \frac{1}{|\mathbb{N}((\alpha))|^{\ell+1/2}} \int_{\mathbf{R}^{r-1}}\frac{dx_1\cdots dx_{r-1}}{\prod_{i=1}^{r-1}(e^{x_i/2}+e^{-x_i/2})}\\
\ll&\sum_{\mathfrak{a}\text{ all integral ideals}} \frac{1}{|\mathbb{N}(\mathfrak{a})|^{\ell+1/2}}<\infty. \qquad \blacksquare
\end{aligned}$$

PROPOSITION 3.14: *Let* $\gamma_0 = \left(\begin{smallmatrix} a & \\ & b\end{smallmatrix}\right)$ *be a hyperbolic element such that* $\sigma_i(ab) > 0$ *for* $i = 1, \ldots, r$. *Let* $\mathfrak{o}$ *be its conjugacy class. Let* $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$. *Then* $k_{\mathfrak{o}}^T(g, f)$ *is absolutely integrable over* $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$ *and*

$$J_{\mathfrak{o}}^T(f) = \mathrm{meas}(F^*\backslash\mathbf{A}^1)\int_{\overline{M}(\mathbf{A})\backslash\overline{G}(\mathbf{A})} f(g^{-1}\gamma_0 g)\Big(T-\frac{1}{2}v(g)\Big)dg, \tag{26}$$

where $v(g) = H(g) + H(wg)$. *The height function* H *is defined few lines before Section 2.1.*

Proof. The proof is divided into two steps:

STEP 1: Arthur's modified truncated kernel function

$$\widetilde{k}_{\mathfrak{o}}^T(g,f) = K(g,g) - \sum_{\delta\in B(F)\backslash G(F)}\sum_{\nu\in N(F)}\sum_{\mu\in\mathfrak{o}\cap\overline{M}(F)} f(g^{-1}\delta^{-1}\mu\nu\delta g)\,\tau(H(\delta g)-T)$$

is absolutely integrable over $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$ and

$$\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} k_{\mathfrak{o}}^T(g,f)dg = \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \widetilde{k}_{\mathfrak{o}}^T(g,f)dg.$$

STEP 2: $\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \widetilde{k}_{\mathfrak{o}}^T(g,f)dg$ is equal to the right hand side of (26). The proof for compactly supported f and $F = \mathbf{Q}$ can be found in [Ar1, Section 8] or [Ar2, Section 11]. The reader is referred to [KL, Section 20] for a proof for $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$ and $F = \mathbf{Q}$. With some obvious modifications, the proof in [KL] can be generalized to other F. ∎

COROLLARY 3.15: *Let $\gamma_0 = \left(\begin{smallmatrix} a & \\ & b \end{smallmatrix}\right)$ be a hyperbolic element such that $\sigma_i(ab) > 0$ for $i = 1, \ldots, r$. Let $\mathfrak{o}$ be its conjugacy class. When $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$, the constant term of $J_{\mathfrak{o}}^T(f)$ is 0.*

Proof. By [KL, Proposition 20.6], the integrals of the constant term and the coefficient of T in (26) are absolutely convergent. By (26), the constant term of $J_{\mathfrak{o}}^T(f)$ is given by

$$-\frac{\text{meas}(F^*\backslash\mathbf{A}^1)}{2} \int_{\overline{M}(\mathbf{A})\backslash\overline{G}(\mathbf{A})} f(x^{-1}\gamma_0 x)v(x)dx. \tag{27}$$

The above integral is called a weighted orbital integral. The orbital integral

$$\begin{aligned}
\int_{\overline{M}(\mathbf{R})\backslash\overline{G}(\mathbf{R})} f_{\mathfrak{k}_i}(g^{-1}\sigma_i(\gamma_0)g)dg &= \int_{\mathbf{R}} f_{\mathfrak{k}_i}\left(\begin{pmatrix} 1 & -t \\ & 1 \end{pmatrix}\begin{pmatrix} \sigma_i(a) & \\ & \sigma_i(b) \end{pmatrix}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}\right)dt \\
&= \int_{\mathbf{R}} f_{\mathfrak{k}_i}\left(\begin{pmatrix} \sigma_i(a) & \\ & \sigma_i(b) \end{pmatrix}\begin{pmatrix} 1 & (1-\frac{\sigma_i(b)}{\sigma_i(a)})t \\ & 1 \end{pmatrix}\right)dt \\
&= 0.
\end{aligned}$$

The last step is from Lemma 2.1. The discussion in [Ge, Lecture V] (especially Proposition 1.1) shows that if the local orbital integrals vanish at 2 distinct places v_1 and v_2, then the the weighted orbital integral (27) vanishes. Taking $v_1 = \infty_1, v_2 = \infty_2$, we can show that the weighted orbital integral is 0. ∎

LEMMA 3.16: *Let $f = f_{\mathfrak{k}} f^{\mathfrak{n}}$.*

$$\sum_{t\in F^*} f\left(g^{-1}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}g\right) - \int_{N(\mathbf{A})} f(g^{-1}ng)\tau(H(g)-T) \tag{28}$$

is absolutely integrable over $\overline{B}(F)\backslash\overline{G}(\mathbf{A})$.

Proof. It suffices to show that (28) is absolutely integrable over $\mathfrak{S}_0$. We follow the proof of [KL, Lemma 21.3]. It remains to show that

$$\sum_{t\in F^*} \left| f\left(g^{-1}\begin{pmatrix} 1 & t \\ & 1 \end{pmatrix}g\right)\right| \tag{29}$$

is absolutely integrable over $\widetilde{\mathfrak{S}}_0^T = \{g \in \mathfrak{S}_0 : H(g) \le T\}$. Because $\mathfrak{S}_{0,\text{fin}}$ is a compact set, it is easy to see that the support of $t \mapsto f_{\text{fin}}(g_{\text{fin}}^{-1}\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)g_{\text{fin}})$ is compact, and hence is contained in $\frac{1}{M}\widehat{\mathcal{O}}$ for some positive integer M. Using the notation in (16), by (8) we have

$$(29) \ll \sum_{t \in \mathcal{O}-\{0\}} \frac{1}{\prod_{i=1}^{r}(4 + y^{-2} m_{\infty_i}^{-2} M^{-2} \sigma_i(t)^2)^{\mathtt{k}_i/2}}$$

We have the following inequalities

$$\frac{1}{4 + y^{-2} m_{\infty_i}^{-2} M^{-2} \sigma_i(t)^2} \leq \frac{M^2 m_{\infty_i}^2 y^2}{\sigma_i(t)^2}, \quad \frac{1}{4 + y^{-2} m_{\infty_i}^{-2} M^{-2} \sigma_i(t)^2} \leq \frac{1}{4}.$$

For $t \in \mathcal{O} - \{0\}$, there exists two different indexes i, j (depending on t) such that $|\sigma_i(t)\sigma_j(t)| \geq 1$. Otherwise $1 > \prod_{i \neq j} |\sigma_i(t)\sigma_j(t)| = |\,\mathrm{N}_{F/\mathbf{Q}}(t)|^{r-1} \geq 1$. This is absurd. We apply the first inequality to i and j. We apply the second inequality to other indexes.

$$\begin{aligned}(29) &\ll \sum_{t \in \mathcal{O}-\{0\}} \frac{y^2}{2^{r-2}} \frac{1}{\prod_{i=1}^{r}(4 + y^{-2} m_{\infty_i}^{-2} M^{-2} \sigma_i(t)^2)^{(\mathtt{k}_i-1)/2}} \\ &\ll y^2 \sum_{t \in \mathcal{O}-\{0\}} \frac{1}{\prod_{i=1}^{r}(4 + e^{-2T} m_{\infty_i}^{-2} M^{-2} \sigma_i(t)^2)^{(\mathtt{k}_i-1)/2}} \\ &\ll y^2 \int_{\mathbf{R}^r} \frac{dx_1 \cdots dx_r}{\prod_{i=1}^{r} (1 + x_i^2)^{(\mathtt{k}_i-1)/2}} \ll y^2.\end{aligned}$$

Therefore,

$$\int_{\widetilde{\mathfrak{S}}_0^T} (29) dg \ll \int_{C_{01}} \int_{C_{02}} \int_0^{e^T} y^2 \frac{dy}{y^2} dm dx < \infty. \qquad \blacksquare$$

Proposition 3.17: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$. When $\mathfrak{o}$ is unipotent,*

(30)

$$J_{\mathfrak{o}}^T(f) = \mathrm{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A})) f(e) + \mathrm{f.\,p.}_{s=1}\, \zeta(\Psi, s) + T \,\mathrm{meas}(F^* \backslash \mathbf{A}^1) \int_{\mathbf{A}} \Psi(t) dt,$$

where e is the identity matrix,

$$\Psi(t) = \int_K f\left(k^{-1} \begin{pmatrix} 1 & t \\ & 1 \end{pmatrix} k\right) dk,$$

and $\mathrm{f.p.}_{s=1}\zeta(\Psi, s)$ *is the finite part at $s = 1$, i.e., the constant term of the Laurent expansion about $s = 1$ of the Tate integral*

$$\zeta(\Psi, s) = \int_{\mathbf{A}^*} \Psi(a)|a|^s d^*a.$$

Proof. When f is a compactly supported function, see [Ge, Lecture IV Proposition 1.2] or [GJ, p. 235–238] for the proof. The proof can be generalized to $f = f_{\mathtt{k}} f^{\mathfrak{n}}$ (see [KL, Chapter 21] for the case of $F = \mathbf{Q}$). By replacing [KL, Lemma 21.3] by Lemma 3.16, the proof in [KL] can be generalized to other F with some obvious modifications. ∎

Proposition 3.18:

$$\zeta(\Psi_{\infty_i}, s) = \int_{\mathbf{R}^*} \int_{K_{\infty_i}} f_{\mathtt{k}_i}(k^{-1}\left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)k)|a|^s dk\, d^*a$$

is absolutely convergent when $0 < \operatorname{Re} s < \mathtt{k}_i$ and hence defines an analytic function for $0 < \operatorname{Re} s < \mathtt{k}_i$. It has a zero at $s = 1$.

Proof. The absolute convergence follows from the bound (8), see [KL, Section 25.1]. The proposition follows easily from Lemma 2.1 by switching the order of the integrals and taking $s = 1$. ∎

Proposition 3.19: *$\zeta(\Psi, s)$ has a zero at $s = 1$. i.e.*

$$\text{f.p.}_{s=1}\zeta(\Psi, s) = 0.$$

Proof. Let $\Psi_v(t) = \int_{K_v} f_v(k^{-1}\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)k)dk$. Let $\Psi_{\text{fin}} = \prod_{v<\infty} \Psi_v$. When $v \nmid \mathfrak{n}$, Ψ_v is the characteristic function of $\mathcal{O}_v$. By Tate's theory, $\zeta(\Psi_{\text{fin}}, s)$ is a meromorphic function with the only possible simple pole $s = 1$ in $\operatorname{Re} s > 0$. By the previous proposition, the order of $\zeta(\Psi, s)$ at $s = 1$ is $r - 1 \geq 1$. ∎

Corollary 3.20: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$. When $\mathfrak{o}$ is unipotent, the constant term of $J^T_{\mathfrak{o}}(x, f)$ is $\operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))f(e)$.*

Proof. This follows easily from the previous proposition and Proposition 3.17. ∎

Theorem 3.21: *Let $f = f_{\mathtt{k}} f^{\mathfrak{n}}$, then*

$$\operatorname{tr} R(f) = \operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))f(e) + \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\gamma \text{elliptic}} f(g^{-1}\gamma g)dg,$$

where e is the identity matrix.

Proof. When $\mathfrak{o}$ is elliptic, $k^T_{\mathfrak{o}}(g, f) = \sum_{\gamma\in\mathfrak{o}} f(g^{-1}\gamma g)$. Thus $\sum_{\mathfrak{o} \text{ elliptic}} k^T_{\mathfrak{o}}(g, f) = \sum_{\gamma \text{ elliptic}} f(g^{-1}\gamma g)$. By Corollary 3.9, the sum is absolutely integrable over $\overline{G}(F)\backslash\overline{G}(\mathbf{A})$.

The number of $a \in \mathcal{O}/\mathcal{O}^*$ such that $a|\mathfrak{n}_i u_j$ is finite. Therefore, by Propositions 3.10 and 3.11, $\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\mathfrak{o}} |k_{\mathfrak{o}}^T(g,f)|dg < \infty$. By (24) and (25),

$$\begin{aligned}\operatorname{tr} R(f) &= \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\mathfrak{o}} k_{\mathfrak{o}}^T(g,f)dg \\ &= \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\gamma \text{ elliptic}} f(g^{-1}\gamma g)dg + \sum_{\mathfrak{o} \text{ hyperbolic or unipotent}} J_{\mathfrak{o}}^T(f).\end{aligned}$$

The right hand side is a linear function of T but the left hand is independent of T. Thus the coefficient of T is zero. The result now follows easily from Corollaries 3.15 and 3.20. ∎

4. The Distribution of Hecke Eigenvalues

Lemma 4.1: *Let $\mathfrak{N}$ be a prime ideal of $\mathcal{O}$. Let w be the corresponding valuation. Suppose $\gamma \in \mathrm{GL}_2(\mathcal{O}_w)$ with $(\det\gamma) = \mathfrak{n}_w$. Let χ_w be the characteristic function of $\overline{M(\mathfrak{n}_w, \mathfrak{N}_w)}$.*

If γ is not conjugate to $\left(\begin{smallmatrix} a & * \\ & a\end{smallmatrix}\right)$ in $M_2(\mathcal{O}_w/\mathfrak{N}_w)$ (when $\mathfrak{N} \nmid 2$, this is equivalent to $(\operatorname{tr}\gamma)^2 - 4\det\gamma \notin \mathfrak{N}_w$), then

$$\int_{K_w} \chi_w(k^{-1}\gamma k)dk \le 2\psi(\mathfrak{N}_w)^{-1}.$$

Proof. The inequality is trivial if the integral is zero. By Lemma 2.3, $\chi_w(k^{-1}\gamma k)$ is nonzero if and only if $k^{-1}\gamma k \in M(\mathfrak{n}_w, \mathfrak{N}_w)$, equivalently $k^{-1}\gamma k \equiv \left(\begin{smallmatrix} * & * \\ & *\end{smallmatrix}\right)$ (mod $\mathfrak{N}_w$). If the above integral is nonzero, then there exists $k_0 \in K_w$ such that $k_0^{-1}\gamma k_0 \equiv \left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)$ (mod $\mathfrak{N}_w$). Notice that $\mathcal{O}_w/\mathfrak{N}_w$ is a finite field. If $a \neq d$ in $\mathcal{O}_w/\mathfrak{N}_w$, we can find $t \in \mathcal{O}_w$ such that $\left(\begin{smallmatrix} 1 & t \\ & 1\end{smallmatrix}\right)^{-1}\left(\begin{smallmatrix} a & b \\ 0 & d\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & t \\ & 1\end{smallmatrix}\right) \equiv \left(\begin{smallmatrix} a & \\ & d\end{smallmatrix}\right)$ (mod $\mathfrak{N}_w$). Therefore, without loss of generality we can assume $b = 0$. A simple calculation shows that $k^{-1}\left(\begin{smallmatrix} a & 0 \\ 0 & d\end{smallmatrix}\right)k \equiv \left(\begin{smallmatrix} * & * \\ & *\end{smallmatrix}\right)$ (mod $\mathfrak{N}_w$) if and only if $k \in K_0(\mathfrak{N}_w)$ or $k \in \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)K_0(\mathfrak{N}_w)$. Thus in this case

$$\begin{aligned}\int_{K_w} \chi_w(k^{-1}\gamma k)dk &= \operatorname{meas}(k_0 K_0(\mathfrak{N}_w)) + \operatorname{meas}\left(k_0\begin{pmatrix} 0 & 1 \\ 1 & 0\end{pmatrix}K_0(\mathfrak{N}_w)\right) \\ &= 2\psi(\mathfrak{N}_w)^{-1}.\end{aligned}$$

This proves the lemma. ∎

Proposition 4.2: *Let $\mathfrak{N}_i$ be a sequence of prime ideals as described in Theorem 1.1. Recall that $f_{\mathfrak{N}_i}^{\mathfrak{n}}$ is the finite part of the test function corresponding to*

$\mathfrak{N} = \mathfrak{N}_i$. *Then*

$$\lim_{i\to\infty} \sum_{\gamma \text{ elliptic}} \int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \psi(\mathfrak{N}_i)^{-1}(f_{\mathtt{k}} \times f^{\mathfrak{n}}_{\mathfrak{N}_i})(g^{-1}\gamma g)dg = 0.$$

Proof. When $g \in \mathfrak{S}'$, by Corollary 2.6 we can replace $\sum_{\gamma \text{ elliptic}}$ by

$$\sum_{\ell,j} \sum_{\substack{\gamma \text{ elliptic},\\ \gamma\in D(\mathtt{n}_\ell u_j, Q)}} .$$

Now

$$\begin{aligned}\int_{\overline{G}(F)\backslash\overline{G}(\mathbf{A})} \sum_{\gamma \text{ elliptic}} |f(g^{-1}\gamma g)|dg &\le \int_{\mathfrak{S}'} \sum_{\gamma \text{ elliptic}} |f(g^{-1}\gamma g)|dg \\ &= \sum_{\ell,j} \sum_{\substack{\gamma \text{ elliptic},\\ \gamma\in D(u_\ell \mathtt{n}_j, Q)}} \int_{\mathfrak{S}'} |f(g^{-1}\gamma g)|dg.\end{aligned}$$

Let χ_{iv} be the characteristic function of $\overline{M(\mathfrak{n}_v, \mathfrak{N}_{iv})}$. Let χ_i be $\prod_{v<\infty} \chi_{iv}$. The proposition follows if we can prove that

$$\lim_{i\to\infty} \sum_{\substack{\gamma \text{ elliptic},\\ \gamma\in D(\mathtt{n}_\ell u_j, Q)}} \int_{\mathfrak{S}'} |(f_{\mathtt{k}} \times \chi_i)(g^{-1}\gamma g)|dg = 0.$$

Since $\mathbb{N}(\mathfrak{N}_i) \to \infty$, we can assume $\mathfrak{N}_i \nmid 2Q\mathfrak{b}_1 \cdots \mathfrak{b}_t$. Let w_i be the valuation corresponding to $\mathfrak{N}_i$. When $\mathfrak{N}_i$ is sufficiently large, $K'_{w_i} = K_{w_i}$. For fixed $\mathtt{n}_\ell$, u_j, partition the elliptic elements $\gamma \in D(\mathtt{n}_\ell u_j, Q)$ into two sets: $S_1 = \{\gamma : (\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma \notin \mathfrak{N}_i\}$ and $S_2 = \{\gamma : (\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma \in \mathfrak{N}_i\}$.

Since $\mathfrak{N}_i \nmid Q$, $\gamma \in M_2(\mathcal{O}_{w_i})$. Since $\mathfrak{N}_i \nmid \mathfrak{b}_\ell$, $\mathtt{n}_\ell\mathcal{O}_{w_i} = \mathfrak{n}_{w_i}$ by (11). Apply the previous lemma and Proposition 3.7 for the first set S_1,

$$\begin{aligned}&\sum_{\gamma\in S_1} \int_{\mathfrak{S}'} |(f_{\mathtt{k}} \times \chi_i)(g^{-1}\gamma g)|dg \\ &= \sum_{\gamma\in S_1} \int_{\mathfrak{S}'_\infty} |f_{\mathtt{k}}|(g_\infty^{-1}\gamma g_\infty)dg_\infty \int_{K_{w_i}} \chi_{iw_i}(g_{w_i}^{-1}\gamma g_{w_i})dg_{w_i} \prod_{v\neq w_i} \int_{K'_v} \chi_{iv}(g_v^{-1}\gamma g_v)dg_v \\ &\le 2\psi(\mathfrak{N}_i)^{-1} \prod_{v\neq w_i} \operatorname{meas}(K'_v) \int_{\mathfrak{S}'_\infty} \sum_{\substack{\gamma \text{ elliptic},\\ \gamma\in D(\mathtt{n}_\ell u_j, Q)}} |f_{\mathtt{k}}|(g_\infty^{-1}\gamma g_\infty)dg_\infty \ll \psi(\mathfrak{N}_i)^{-1}.\end{aligned}$$

Notice that $\psi(\mathfrak{N}_i)^{-1} \to 0$ when $i \to \infty$.

For the second set S_2, let a be an integer ≥ 1. When γ is elliptic, $(\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma \neq 0$. Therefore, $|\operatorname{N}_{F/\mathbf{Q}}((\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma)|$ is a positive integer. Consider the following summation

$$B_a = \sum_{\substack{\gamma \text{ elliptic}, \gamma \in D(\mathfrak{n}_\ell u_j, Q), \\ |\mathrm{N}_{F/\mathbf{Q}}((\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma)| = a}} \int_{\mathfrak{S}'_\infty} |f_{\mathrm{k}}|(g^{-1}\gamma g)dg.$$

From Proposition 3.7, $\sum_{a\geq 1} B_a$ is convergent. Therefore, $\sum_{a\geq A} B_a \to 0$, when $A \to \infty$.

When $\gamma \in S_2$, $\mathbb{N}(\mathfrak{N}_i) | \mathrm{N}_{F/\mathbf{Q}}((\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma)$. Thus

$$|\mathrm{N}_{F/\mathbf{Q}}((\operatorname{tr} Q\gamma)^2 - 4\det Q\gamma)| \geq \mathbb{N}(\mathfrak{N}_i).$$

We have

$$\left| \sum_{\gamma \in S_2} \int_{\mathfrak{S}'} (f_{\mathrm{k}} \times \chi_i)(g^{-1}\gamma g)dg \right| \leq \operatorname{meas}(K'_{\mathrm{fin}}) \sum_{a \geq \mathbb{N}(\mathfrak{N}_i)} B_a \to 0 \text{ when } i \to \infty.$$

This completes the proof. ∎

Proposition 4.3: *Let $\mathfrak{n} = \mathfrak{p}^n$, then*

$$\lim_{i\to\infty} \frac{\sum_{\pi \in \Pi_{\mathrm{k}}(\mathfrak{N}_i)} X_n(\lambda_v(\pi))}{|\Pi_{\mathrm{k}}(\mathfrak{N}_i)|} = \begin{cases} q_v^{-n/2} & \text{if } 2|n, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Let $\mathfrak{N} = \mathfrak{N}_i$. Let $\chi_{\mathfrak{N}}^{\mathfrak{n}}$ be the characteristic function of $\prod_{v<\infty} \overline{M(\mathfrak{n}_v, \mathfrak{N}_v)}$. Using the trace formula in Theorem 3.21 and the previous proposition, we have

$$\operatorname{tr} R(f_{\mathrm{k}} f_{\mathfrak{N}}^{\mathfrak{n}}) = \operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))\psi(\mathfrak{N}) f_{\mathrm{k}}(e)\chi_{\mathfrak{N}}^{\mathfrak{n}}(e) + o(\psi(\mathfrak{N})). \tag{31}$$

Obviously, $\operatorname{tr} R(f_{\mathrm{k}} f_{\mathcal{O}}^{\mathfrak{n}}) = O(1) = o(\psi(\mathfrak{N}))$. Therefore, by Proposition 3.6, we have

$$q_v^{n/2} \sum_{\pi \in \Pi_{\mathrm{k}}(\mathfrak{N})} X_n(\lambda_v(\pi)) = \operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))\psi(\mathfrak{N}) f_{\mathrm{k}}(e)\chi_{\mathfrak{N}}^{\mathfrak{n}}(e) + o(\psi(\mathfrak{N})) \tag{32}$$

Following the argument in Proposition 2.4, we can show that $\chi_{\mathfrak{N}}^{\mathfrak{n}}(e) = 1$ only if there exists an ideal $\mathfrak{b}$ such that $\mathfrak{b}^2\mathfrak{n} = (1)$, i.e., n is even. Under this assumption, $\mathfrak{b} = \mathfrak{p}^{-n/2}$ and $\chi_{\mathfrak{N}}^{\mathfrak{n}}(e) = 1$ if and only if $\mathfrak{b}_w^{-1} e = \mathfrak{p}_w^{n/2} e \subset M(\mathfrak{n}_w, \mathfrak{N}_w)$ for all finite places w, which is automatically true. Therefore,

$$\chi_{\mathfrak{N}}^{\mathfrak{n}}(e) = \begin{cases} 1 & \text{if } 2|n, \\ 0 & \text{otherwise.} \end{cases}$$

Taking $n = 0$ in (32), we have

$$|\Pi_{\mathrm{k}}(\mathfrak{N})| = \operatorname{meas}(\overline{G}(F)\backslash\overline{G}(\mathbf{A}))\psi(\mathfrak{N}) f_{\mathrm{k}}(e) + o(\psi(\mathfrak{N})). \tag{33}$$

The proposition follows easily by taking the quotient of (32) and (33). ∎

Proof of theorem 1.1: From [Se1, (17)],

$$\frac{q_v+1}{(q_v^{1/2}+q_v^{-1/2})^2-x^2}=\sum_{m=0}^{\infty}q_v^{-m}X_{2m}(x).$$

By the orthonormality of $\{X_m\}$ relative to $d\mu_\infty$ [Se1, (16)] and the previous proposition, we have

$$\lim_{i\to\infty}\frac{\sum_{\pi\in\Pi_{\mathrm{k}}(\mathfrak{N}_i)}X_n(\lambda_v(\pi))}{|\Pi_{\mathrm{k}}(\mathfrak{N}_i)|}=\int_{\mathbf{R}}X_n(x)d\mu_v(x). \tag{34}$$

The values $\lambda_v(\pi)$ are contained in a finite interval I_v [Ro, Proposition 2.9]. We can take $I_v=[-2,2]$ if we assume the validity of the Ramanujan conjecture but we do not need this strong result. Because polynomials are dense in $L^\infty(I_v)$, (34) remains true if we replace X_n by an arbitrary continuous function. This proves the main theorem. ∎

References

[Ar1] J. Arthur, *A trace formula for reductive groups. I.* Duke Mathematical Journal **45** (1978), 911–952.

[Ar2] J. Arthur, *An introduction to the trace formula, Harmonic analysis, the trace formula, and Shimura varieties*, in *Clay Mathematics Proceedings*, vol. 4, American Mathematical Society, Providence, RI, 2005, pp. 1–263.

[BJ] A. Borel and H. Jacquet, *Automorphic forms and automorphic representations*, in *Proceedings of Symposia in Pure Mathematics*, vol. 33, Part 1, American Mathematical Society, Providence, RI, 1979, pp. 189-207.

[Bl] Blasius, D., *Hilbert modular forms and the Ramanujan conjecture,* in *Noncommutative Geometry and number Theory*, Vieweg Verlag, Wiesbaden, 2006, pp. 35–56.

[BL] J.-L. Brylinski and J.-P. Labesse, *Cohomologie d'intersection et fonctions L de certaines variétés de Shimura,* Annales Scientifiques de l'École Normale Supérieur, Quatrième Série **17** (1984), 361–412.

[Bu] D. Bump, *Automorphic Forms and Representations*, Cambridge Studies in Advanced Mathematics **55**, Cambridge University Press, Cambridge, 1998.

[Ca] W. Casselman, *On some results of Atkin and Lehner*, Mathematische Annalen **201** (1973), 301–314.

[CDF] B. Conrey, W. Duke and D. Farmer, *The distribution of the eigenvalues of Hecke operators,* Acta Arithmetica **78** (1997), 405–409.

[De] P. Deligne, *Formes modulaires et répresentations l-adiques*, Séminaire Bourbaki 1968/69, exposé 347–367, Lecture Notes in Mathematics, vol. 179, Spring-Verlag, Berlin, 1971, pp. 139–172.

[DS] P. Deligne and J.P. Serre, *Formes modulaires de poids* 1, Annales Scientifiques de l'École Normale Supérieur, Quatrième Série **7** (1974), 507–530 (1975).

[Fl] D. Flath, *Decomposition of representations into tensor products,* Proceedings of Symposia in Pure Mathematics, vol. 33, Part 1, American Mathematical Society, Providence, RI, 1979, pp. 179–183.

[Ge] S. Gelbart, *Lectures on the Arthur-Selberg Trace Formula*, American Mathematical Society, Providence, RI, 1996.

[GJ] S. Gelbart and H. Jacquet, *Forms of* GL(2) *from the Analytic Point of View*, Proceedings of Symposia in Pure Mathematics, vol. 33, American Mathematical Society, Providence, RI, 1979.

[GGPS] I. M. Gel'fand, M. I. Graev and I. I. Piatetski-Shapiro, *Representation Theory and Automorphic Functions*, W. B. Saunders Co., 1969.

[Go] R. Godement, *Domaines Fondamentaux des Groupes Arithmétiques*, Séminaire Bourbaki, 1962/63. Fasc. 3, vol. 257. Secrétariat mathématique, Paris, 1964.

[KL] A. Knightly and C. Li, *Traces of Hecke Operators*, Mathematical Surveys and Monographs series, vol. 133, American Mathematics Society, 2006.

[La] S. Lang, $\mathrm{SL}_2(\mathbf{R})$, Springer-Verlag, New York, 1985.

[Mu] R. Murty, *Applications of Symmetric Power L-functions*, in *Lectures on automorphic L-functions*, Fields Institute Monographs. , vol. 20, American Mathematical Society, Providence, RI, 2004, pp. 203–283.

[Ng] L. Nguyen, *The Ramanujan conjecture for Hilbert modular forms,* Ph.D. thesis, UCLA, 2005.

[Os] M. S. Osborne, *Spectral Theory and Uniform Lattices*, Lecture notes in representation theory, University of Chicago Dept. of Mathematics, 1977.

[Ro] J. Rogawski, *Modular forms, the Ramanujan conjecture and the Jacquet-Langlands correspondence*, appendix in "Discrete Groups, Expanding Graphs and Invariant Measures," by A. Lubotzky, Birkhäuser, Basel, 1994, pp. 135-176.

[Sa] P. Sarnak, *Statistical properties of eigenvalues of the Hecke operators*, Analytic number theory and Diophantine problems (Stillwater, OK, 1984), Progress in Mathematics, vol. 70, Birkhuser Boston, Boston, MA, 1987, pp. 321–331.

[Se1] J. P. Serre, *Répartition Asymptotique des valeurs propres de l'opérateur de Hecke T_p,* Journal of the American Mathematical Society **10** (1997), 75–102.

[Se2] J. P. Serre, *Abelian l-adic Representations and Elliptic Curves*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[Sh] F. Shahidi, *Symmetric power L-functions for* GL(2), in *Elliptic curves and related topics*, CRM Proc. Lecture Notes, vol. 4, American Mathematical Society, Providence, RI, 1994, pp. 159–182.

[Ta] R. Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations II,* preprint.